Arrangements of Geodesic Arcs on the Sphere*

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ABSTRACT

This movie illustrates exact construction and maintenance of arrangements induced by arcs of great circles embedded on the sphere, also known as geodesic arcs, and exact computation of Voronoi diagrams on the sphere, the bisectors of which are geodesic arcs. This class of Voronoi diagrams includes the subclass of Voronoi diagrams of points and its generalization, power diagrams, also known as Laguerre Voronoi diagrams. The resulting diagrams are represented as arrangements, and can be passed as input to consecutive operations supported by the `Arrangement_2` package of CGAL and its derivatives. The implementation handles well degenerate input and produces exact results.

Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*Geometrical problems and computations*

General Terms

Algorithms, Experimentation, Performance

1. INTRODUCTION

Given a finite collection $\mathcal{C}$ of geometric objects (such as lines, planes, or spheres) the *arrangement* $\mathcal{A}(\mathcal{C})$ is the subdivision of the space where these objects reside into cells as induced by the objects in $\mathcal{C}$. In this movie we concentrate on the particular class of arrangements, where the embedding space is the sphere, and the inducing objects are geodesic arcs. There is an analogy between this class of arrangements and the class of planar arrangements induced by linear curves (i.e., segments, rays, and lines), as properties of linear curves in the plane can be often, but not always, adapted to geodesic arcs on the sphere. The ability to robustly construct arrangements of geodesic arcs on the sphere, and carry out exact operations on them using only (exact) rational arithmetic is a key property that enables an efficient implementation.

The `Arrangement_2` package of CGAL, the Computational Geometry Algorithms Library,[1] has recently been extended to support arrangements of curves embedded on two-dimensional parametric surfaces [2]. The package was used to compute arrangements on quadrics [2] and on Dupin cyclides [3], which contain the torus as a special case.

The concept of computing cells of points that are closer to a certain object than to any other object, among a finite number of objects, was extended to various kinds of geometric sites, ambient spaces, and distance functions, e.g., power diagrams of circles in the plane, multiplicatively weighted Voronoi diagrams, additively weighted Voronoi diagrams [1]. This space decomposition is strongly connected to arrangements [4], a property that yields a very general approach for computing Voronoi diagrams. One immediate extension is computing Voronoi diagrams on two-dimensional parametric surfaces in general, and on the sphere in particular.

2. ARRANGEMENTS ON SURFACES

A parameterized surface S is defined by a function $f_S : \mathbb{P} \rightarrow \mathbb{R}^3$, where the domain $\mathbb{P} = U \times V$ is a rectangular two-dimensional parameter space with bottom, top, left, and right boundaries, and the range f_S is a continuous function. We allow $U = [u_{\min}, u_{\max}]$, $U = [u_{\min}, +\infty)$, $U = (-\infty, u_{\max}]$, or $U = (-\infty, +\infty)$, and similarly for V. A *contraction point* $p \in S$ is a singular point, which is the mapping of a whole boundary of the domain $\mathbb{P}$. An *identification curve* $C \subset S$ is a continuous curve, which is the mapping of opposite closed boundaries of the domain $\mathbb{P}$.

The extended `Arrangement_2` package handles infinite curves as well as curves that reach the boundaries. A boundary may define a contraction point or an identification curve.

The `Arrangement_2` package offers various operations on arrangements of curves, such as point location, insertion of curves, removal of curves, and overlay computation. All steps of these operations are enabled by a minimal set of geometric primitives, such as comparing two points in uv-lexicographic order, computing intersection points, etc. These primitives are gathered in a traits class, which models a

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[1] http://www.cgal.org

geometry-traits [8]. Different geometry-traits classes are provided in the `Arrangement_2` package to handle various families of curves, e.g., line segments, conic arcs, etc.

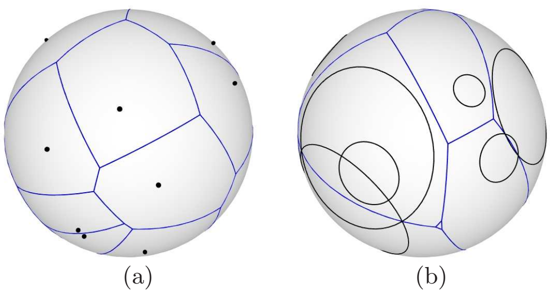
3. APPLICATIONS

We use the following parameterization of the unit sphere: $\mathbb{P} = [-\pi, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, $f_S(u, v) = (\cos u \cos v, \sin u \cos v, \sin v)$. This parameterization induces two contraction points $p_s = (0, 0, -1)$ and $p_n = (0, 0, 1)$, referred to as the south and north poles respectively, and an identification curve that coincides with the opposite Prime (Greenwich) Meridian.

The implementation of the geometry-traits class for geodesic arcs on the sphere handles all degeneracies, and is exact as long as the underlying number type supports the arithmetic operations $+$, $-$, $*$, and $/$ in unlimited precision over the rationals, such as the one provided by GMP[2]. An advantage of our implementation is that all the required geometric operations listed in the traits concept are implemented using only rational arithmetic.

Armed with the geometry traits for geodesic arcs on the sphere, we compute Minkowski sums of convex polyhedra, by overlaying their respective Gaussian maps, which are arrangements of geodesics on the sphere. We also compute two types of Voronoi diagrams on the sphere through the computation of the lower envelope of the site-distance functions over the sphere. We define lower envelopes of functions on the sphere in a way similar to the standard definition of lower envelopes of bivariate functions in space.

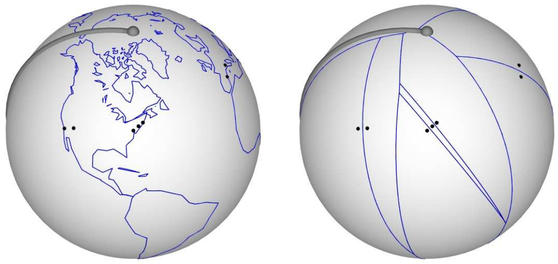
A new framework based on the envelope algorithm of CGAL [6] was developed to compute different types of Voronoi diagrams. The framework provides a reduced and convenient interface between the construction of the diagrams and the construction of envelopes. Obtaining a new type of Voronoi diagrams requires the provision of a traits class that handles the type of bisector curves of the new diagram type [5]. Essentially, every type of Voronoi diagram, the bisectors of which can be handled by an arrangement traits class, can be implemented using this framework. The bisector curves between point sites on the sphere are great circles, handled by the newly developed traits class, and so are the bisectors between circle sites on the sphere under the Laguerre distance [7]; see Figure 1.



(a) (b)

Figure 1: Voronoi diagrams on the sphere. (a) The Voronoi diagram of 14 random points. (b) The power diagram of 10 random circles.

Figure 2(a) shows an arrangement on the sphere induced by (i) the continents and some of the islands on earth, and (ii) the institutions that hosted SoCG during this millennium, which appear as isolated vertices. The sphere is oriented such that College Park, MD, USA is at the center. The arrangement consists of 1054 vertices, 1081 edges, and 117 faces. The data was taken from gnuplot[3] and google maps[4].



(a) (b)

Figure 2: Arrangements on the sphere.

Figure 2(b) shows an arrangement that represents the Voronoi diagram of the nine cities, the institutions above are located at, namely College Park, Gyeongju, Sedona, Pisa, New York, San Diego, Barcelona, Medford, and Hong Kong. The figure on the right shows the *overlay* of the two arrangements shown in Figure 2.

The individual images that comprise the movie were produced by a program that visualizes 3D objects stored in an extended VRML format[5]. The format was extended with several geometry nodes that represent various types of arrangements embedded on the sphere. Additional material is available at `http://www.cs.tau.ac.il/~efif/VOS`.

4. REFERENCES

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[2] `http://www.swox.com/gmp/`

[3] `http://www.gnuplot.info/`

[4] `http://maps.google.com/`

[5] `http://www.web3d.org/`