Algorithmic Robotics and Motion Planning

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Translational Motion and Minkowski Sums

Dan Halperin
School of Computer Science
Tel Aviv University
Translational motion of a polygon among polygons in the plane

• Very well understood and having efficient complete implementation (which is not the case for almost any other non-trivial MP problem)
• The structure behind the solutions: Minkowski sums
• A few theoretical problems remain open
Today’s lesson

• The combinatorial complexity of polygonal Minkowski sums in the plane
• The connection between motion planning and Minkowski sums
• Algorithms
• Going up to 3D

As time permits:
• More applications of Minkowski sums
• Minkowski average
Polygonal Minkowski sums in the plane

Structure and combinatorial complexity
The Minkowski sum of two sets $P$ and $Q$ in Euclidean space is the result of adding every point in $P$ to every point in $Q$

$$\{(x_1, y_1)\} \oplus \{(x_2, y_2)\} = \{(x_1 + x_2, y_1 + y_2)\}$$
Convex polygons

• The farthest point of the sum in any direction is the sum of the farthest points in that direction of the summands
• The sum of convex polygons is a convex polygon
• Given polygons with $m$ and $n$ vertices, the sum has at most $m + n$ vertices

• Later, an interesting property of sums of convex polygons: pseudodiscs
Non-convex polygons

- Triangulate each polygon, construct the union of sums of pairs of triangles
- Arrangements of segments
- Number of segments $O(mn)$
- Maximum complexity $O(m^2n^2)$
- The bound is tight
So far

Given two polygons with $m$ and $n$ vertices
- Convex – convex: $O(m + n)$
- Non-convex – non-convex: $O(m^2n^2)$

How about convex – non-convex?
One of the early surprising results in CG
Reminder: The union of discs

• Given $m > 2$ discs in the plane, the boundary of the union of regions that they enclose contains at most $6m - 12$ intersection points of the arcs
• This bound is tight
Pseudodiscs

Definition (A pair of pseudodiscs)

A pair of two planar connected point sets $o_1$ and $o_2$ is called a pair of pseudodiscs if $\partial o_1 \cap \text{int}(o_2)$ is connected and $\partial o_2 \cap \text{int}(o_1)$ is connected.

- The boundaries $\partial o_1$ and $\partial o_2$ intersect in at most two points.

![Diagram of pseudodiscs and non-pseudodiscs](image-url)
Convex – non-convex, polygons:
The pseudodiscs property

**Theorem**

Let $P$ and $Q$ be two convex polygons that are interior disjoint, and let $R$ be another convex polygon. The two Minkowski sums $P \oplus R$ and $Q \oplus R$ are pseudodiscs.
Convex – non-convex, polygons: The complexity of the sum

Theorem: the complexity of the Minkowski sum of a convex polygon with $m$ vertices and a simple polygon with $n$ vertices is $O(mn)$

- The bound is tight
General result: The union of pseudodiscs

• Given $m$ simple Jordan curves in the plane, each pair of which intersect one another in at most two points, then the boundary of the union of regions that they enclose contains at most $\max(2, 6m - 12)$ intersection points of the curves, and this bound cannot be improved

[kedem-Livne-Pach-Sharir ‘86]
Minkowski sums and translational motion planning
Why are Minkowski sums so useful?
Here’s a major reason:

• Claim: Two sets $A$ and $B$ intersect if and only if the Minkowski sum $A \oplus -B$ contains the origin, where $-B$ is the set $B$ reflected through the origin

• More generally: $A \cap (B \oplus \{t\}) \neq \emptyset$ iff $t \in A \oplus -B$

In the plane $-B$ is $B$ rotated by $\pi$ radians around the origin
Example: motion planning (piano movers)

- $R$ - a polygonal object that moves by translation
- $P$ - a set of polygonal obstacles

Claim: When translating, $R$ intersects $P$ iff $\text{ref}(R)$ is inside $P \oplus - R$
Algorithms
Algorithms

- Convex - convex
- The general case
  - Representation
  - Decomposition
  - The mystery of the construction time
  - Convolution
  - The hole filter
- Convex – non-convex
Convex – convex

• Merging of normal diagrams
• $O(m + n)$
The general case
Construction by decomposition

• We alluded to it when we gave the general bound

• **Step 1** Decompose $P$ and $Q$ into convex subpolygons $P_1, ..., P_s$ and $Q_1, ..., Q_t$

• **Step 2** Compute $R_{ij} := P_i \oplus Q_j$ for each pair

• **Step 3** Construct the union of those subsums
How to represent Minkowski sums?
The language of arrangements

• Much more involved than the convex case
• Should allow for complex topology, holes of any dimension
• Arrangements of curves and surfaces do the job
Representation of the free space
Vertical decomposition + connectivity graph

[www.seas.upenn.edu/~jwk/motionPlanning]
Representation of the free space
Example
Constructing the union of the subsums
One possibility: the arrg algorithm

• Let $R$ be the set of all $R_{ij}$s
• Add all the edges of $R$ into a planar arrangement
• Compute carefully for each face, edge and vertex whether it is inside union
• Time: $O((I + k) \log k)$
  
  $O(I + k)$ - traversal
  
  $k$ - number of edges in $R$
  
  $I$ - number of intersections among edges of $R$
The mystery of the construction time
Construction by decomposition, zooming in

• **Step 1** Decompose $P$ and $Q$ into convex subparts $P_1, \ldots, P_S$ and $Q_1, \ldots, Q_t$

• **Step 2** Compute $R_{ij} := P_i \bigoplus Q_j$ for each pair

• **Step 3** Construct the union of those subsums

For example: $P$ and $Q$ are polygons with $m + 2$ and $n + 2$ vertices resp.

• **Step 1** Decompose $P$ and $Q$ into $m$ and $n$ triangles

• **Step 2** Compute the hexagon $R_{ij} := P_i \bigoplus Q_j$ for each pair

• **Step 3** Construct the union of those $4mn$ triangles
Algorithm complexity

• Recall that for arbitrary polygons the maximum complexity is $O(m^2n^2)$
• The output may have size $\Omega(m^2n^2)$
• We know to approach this running time in the worst case
• Can we have a guaranteed output-sensitive algorithm?
• Can we efficiently decide if the Minkowski sum has holes?
Hardness in P

- The curious incident of 3-SUM hard problems

**Definition (3SUM)**

Given a set $S$ of $n$ integers, are there elements $a, b, c \in S$ such that $a + b + c = 0$?

- Computing the union of a set of triangles is 3SUM hard
- We need to compute the union of $4mn$ triangles
- HOWEVER, our $4mn$ triangles are special
- The mystery remains

[Gajentaan-Overmars ’95], [Grønlund-Pettie ‘14]
The hole filter

in theory and practice
Convolution

- \( P \otimes Q \) — the convolution of \( P \) and \( Q \) is a collection of line segments:
  - \([p_i + q_j, p_{i+1} + q_j]\), where \( \overrightarrow{p_ip_{i+1}} \) lies between \( q_{j-1}q_j \) and \( q_jq_{j+1} \) and
  - \([p_i + q_j, p_i + q_{j+1}]\), where \( \overrightarrow{q_jq_{j+1}} \) lies between \( \overrightarrow{p_{i-1}p_i} \) and \( \overrightarrow{p_ip_{i+1}} \).

[Guibas-Ramshaw-Stolfi ‘83]
Construction by convolution

• Lemma: The boundary of the sum is included in the convolution of the boundaries
• Step 1 Track the boundaries simultaneously systematically to create a 2D arrg
• Step 2 Compute winding numbers of faces in the arrg
• Step 3 Construct the union of positive winding-number faces
Speeding up the convolution algorithm in practice by filters

• Construct a partial version of the underlying arrangement
  • Pro: fewer edges $\rightarrow$ faster arrangement construction
  • Con: we lose the winding number property

• Apply filters; for example, the reflex vertex filter: reflex vertices do not contribute to the Minkowski sum boundary [Kaul-O’Connor-Srinivasan ‘91]

• Check for each face in the resulting arrangement whether it is inside the sum

[Behar-Lien ‘11]
The hole filter

Q: Given two polygons-w/h, which holes can you fill up and still get the same Minkowski sum?
The hole filter

Q: Given two polygons-w/h, which holes can you fill up and still get the same Minkowski sum?
The hole filter, cont’d

Theorem: Let $H$ be a hole in $P$. Then

$P \oplus Q \neq (P \cup H) \oplus Q$ iff $\exists t \in \mathbb{R}^2$ s.t. $Q \oplus \{t\} \subseteq -H$. 
Easily computable filters

• Check if the hole $H$ in $P$ should be filled up by comparing the axis-aligned bounding box of $H$ and the axis-aligned bounding box of $Q$
The hole filter, cont’d

Corollary:
One can fill up all the holes of at least one polygon and still get the same Minkowski sum
Hole theorem, proof

• Recall that $A \cap (B \oplus \{t\}) \neq \emptyset$ iff $t \in A \oplus -B$
• $(\exists t \in R^2 ...)$ then
  $-Q \oplus \{t\} \subseteq H$
  $P \cap (-Q \oplus \{t\}) = \emptyset$, $(P \cup H) \cap (-Q \oplus \{t\}) \neq \emptyset$
  $t \not\in P \oplus Q$, $t \in (P \cup H) \oplus Q$
• $(\forall t \in R^2 ...)$ then
  $\forall t$, if $(-Q \oplus \{t\}) \cap H \neq \emptyset$ then $(-Q \oplus \{t\}) \cap \partial H \neq \emptyset$, namely $(-Q \oplus \{t\}) \cap P \neq \emptyset$
  $t \in (P \cup H) \oplus Q \Rightarrow t \in P \oplus Q$

\[\square\]
Convex - general

• We saw that when the robot is a convex polygon, the complexity of the free space (complement of the C-obstacles) is favorable: how about algorithms?

• Standard approach: divide-and-conquer, where the merge step uses sweep line to compute the union of two subsets of expanded obstacles

• More efficient approach using medial axis?
More on the decomposition approach

• Variations based on
  • Convex decomposition
  • Union algorithm
Speeding up the decomposition algorithm
Decomposition length effect: an example

P - fixed size, two types of decompositions

Q - fixed decomposition, scaled size
Decomposition length effect: results

Q grows

time for computing the Minkowski sum of a knife polygon $P$ (using two types of decompositions) with a random polygon $Q$ that is scaled differently
Smaller number of intersections of segments

• We want each edge of $R$ to intersect as few polygons of $R$ as possible
• $\mu(L(R_{ij}))$ - the standard rigid-motion invariant measure of the set of lines intersecting $R_{ij}$
• $\mu(L(R_{ij}))$ is the perimeter of $R_{ij}$
Length vs. number of intersections
Optimizing the mixed objective function

\[ k_Q(2\Delta_P + \Pi_P) + k_P(2\Delta_Q + \Pi_Q) \]

- \( k_P \) - number of subpolygons in the convex decomposition of \( P \)
- \( \Delta_P \) - total length of diagonals in the decomposition of \( P \)
- \( \Pi_P \) - the perimeter of \( P \)

The function measures the overall length of the edges of \( R \)

An \( O(n^2r_P^4 + m^2r_Q^4) \)-time decomposition algorithm that minimizes this function

- \( r_P \) - number of reflex vertices in \( P \)

[Agarwal-Flato-H ‘02]
Implementation
The CGAL Minkowski_sum_2 Package

• Based on the Arrangement_2, Polygon_2, and Partition_2 packages
• Our reduced convolution (RC) includes the recent filters: Holes I (the hole theorem) and Holes II (each hole of the sum is part of the convolution of one boundary component from each summand)

Legend in the next slide
• The consumption time in seconds as a function of the # of vertices
• MS of pairs of polygons with holes with $n$ vertices and $n/10$ holes
• More results in the paper

[Baram-Fogel-H-Hemmer-Morr]
3D
Convex polytopes and spherical arrangements

[Berberich-Fogel-H-Setter]
Convex polytopes

• Recall that for polygons with $m$ and $n$ vertices, the sum has at most $m + n$ vertices.

• For polytopes (3D) with $m$ and $n$ vertices, the sum has $\Theta(mn)$ vertices; exact numbers [Fogel-H-Weibel ‘09].
Arbitrary polyhedra

• Recall that for polygons with \( m \) and \( n \) vertices: \( O(m^2n^2) \)
• Polyhedra with \( m \) and \( n \) vertices: \( O(m^3n^3) \)
• These bounds are tight
Minkowski sums, more applications

• Minimum separation distance (penetration depth)
• Placement
• Tolerancing, offsetting
• Nesting
• Cartographic generalization
Assembly planning

the splitStar puzzle

projection of Minkowski sums onto the sphere

[Fogel-H ‘13]
Self-driving cars

[Kleinbort-van den Berg-H]
Standard operations

• Minimum separation distance (penetration depth)

• The Separating Axis Theorem: Given two boxes in 3D (OBBs), one can decide if they collide by testing their projection along 15 lines [Gottschalk et al. ‘96]

• Which are these 15 lines and why are 15 sufficient?
  • Consider the Minkowski sum of the two OBBs
Minkowski average, riddle

Let $A$ be a regular polyhedral set in $\mathbb{R}^d$.

Consider the sequence $A, \frac{A \oplus A}{2}, \frac{A \oplus A \oplus A}{3}, ...$

What can we say about $\frac{A \oplus A \oplus ... \oplus A}{k}$, where $A$ appears $k$ times in the numerator, as $k$ goes to infinity?
Minkowski average, solution

[Shapley-Folkmann-Starr ‘69]
Minkowski average, convergence

Consider the sequence $A, \frac{A \oplus A}{2}, \frac{A \oplus A \oplus A}{3}, \ldots$ in $R^d$.
Does the volume monotonically increase?

For $d = 1$, yes
For $d \geq 12$, no
For $1 < d < 12$, ?

[Fradelizi et al, ‘16]
Open problems and challenges
The mystery of the construction time
Challenges

• Quasi output-sensitive algorithms
• More filters: Given $P$ and $Q$, what is the family of $P'$'s such that $P' ⊕ Q = P ⊕ Q$
Major engineering challenge:
Exact and efficient implementation of the general 3D case

• The decomposition method [Hachenberger ‘09]
• 3D arrangements
  • Collins decomposition
  • Decomposition-sensitive sweep [H-Shaul ‘02]
  • Central difficulty: Degeneracies
  • Geometric Rounding

• Alternative approaches: Lien et al, Manocha et al
THE END