Construction of General Two-Dimensional Voronoi Diagrams via Divide and Conquer Algorithm of Envelopes

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Voronoi Diagrams

- Given $n$ objects (Voronoi sites) in some space (e.g., $\mathbb{R}^d$, $\mathbb{S}^d$) and a distance function $\rho$
- The **Voronoi Diagram** subdivides the space into cells
- Each cell consists of points that are closer to one particular site than to any other site

- Variants include different:
  - Classes of sites
  - Embedding spaces
  - Distance functions

Fractals from Voronoi diagrams
http://www.righto.com/fractals/vor.html
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Voronoi diagram of segments
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Sample of types of Voronoi diagrams

Apollonius Diagram — Diagram of Disks

Apollonius diagram

Given a set \( D = \{d_1, \ldots, d_n\} \) of disks with respective centers \( p_i \) and radii \( r_i \), the diagram is defined by the following distance function:

\[
\rho(x, d_i) = \|x - p_i\| - r_i
\]
Sample of types of Voronoi diagrams

Weighted Diagrams

- Apollonius diagrams are actually additively-weighted Voronoi diagram
Sample of types of Voronoi diagrams

Weighted Diagrams

- Apollonius diagrams are actually additively-weighted Voronoi diagram
- Another type of Voronoi diagrams are the multiplicatively-weighted Voronoi diagrams, or their generalization Möbius diagrams

Möbius diagram

Let $w_i$ be a Möbius site defined by a triple $(p_i, \lambda_i, \mu_i)$ where $p_i \in \mathbb{R}^2$ and $\lambda_i, \mu_i \in \mathbb{R}$. The diagram is defined by the following distance function:

$$\rho(x, w_i) = \lambda_i (x - p_i)^2 - \mu_i$$
Sample of types of Voronoi diagrams
On the Sphere and others
Sample of types of Voronoi diagrams
On the Sphere and others

Other types include:

- Taxi-driver (Manhattan) distance
- Moscow (Karlsruhe) distance
Sample of types of Voronoi diagrams

Farthest-site Voronoi diagrams

Farthest Standard Voronoi Diagram

Farthest Apollonius diagram

Farthest Möbius Diagram

Farthest Diagram of Segments
Applications

- Knuth’s Post-Office problem
- Largest empty circle and other proximity problems.
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- Find the most similar object in a database
- Motion planning — Finding clear paths
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- Find the most similar object in a database
- Motion planning — Finding clear paths
- Epidemiology
  Voronoi diagrams were used to analyze the 1854 cholera epidemic in London. A correlation between deaths and proximity to a particular water pump was discovered
- Minimum-width annulus — to come...
- Astronomy, Climatology, Computer Graphics, Chemistry, Architecture, Geology, Zoology, and more
Outline

1. Voronoi diagrams — Lower Envelopes Connection
2. Divide-and-Conquer Algorithm for Computing Voronoi Diagrams
3. Examples of Selected Voronoi Diagrams
   - Affine Voronoi Diagrams
   - Möbius Diagrams
   - Apollonius Diagrams
4. Application: Minimum-Width Annulus
   - Applications
   - Related Work
   - Proof
Definition

Given a set of bivariate functions \( S = \{s_1, \ldots, s_n\} \), their lower envelope is defined to be their pointwise minimum:

\[
\psi(x, y) = \min_{1 \leq i \leq n} s_i(x, y)
\]

Ophir Setter (TAU)

VD construction via Lower Envelopes

ACG 2009 6 / 34
**Definition**

Given a set of bivariate functions $S = \{s_1, \ldots, s_n\}$, their **lower envelope** is defined to be their pointwise minimum:

$$
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$$

**Corollary**

*Voronoi diagrams are the minimization diagrams (planar projections of the lower envelopes) of the distance functions from each site [Edelsbrunner-Seidel '86]*
Abstract Voronoi Diagrams

- The distance function only tells us the distance from a point.
- Voronoi diagrams can be equivalently defined in terms of their bisectors.
- The bisector $B(o_i, o_j)$ of two sites is the locus of all points that have an equal distance to both sites.
Abstract Voronoi Diagrams

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**Definition**

The abstract Voronoi diagram is defined in terms of bisector and dominance regions (partial definition)

$$\text{Reg}(o_i, O) = \bigcap_{o_j \in O, j \neq i} \text{Reg}(o_i, o_j)$$
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**Definition**

The abstract Voronoi diagram is defined in terms of bisector and dominance regions (partial definition)

$$\text{Reg}(o_i, O) = \bigcap_{o_j \in O, j \neq i} \text{Reg}(o_i, o_j)$$

- The bisector is the projected intersection of the distance functions
- What happens if you invert the dominance regions?
The Divide-and-Conquer Algorithm

Let $S$ be a set of $n$ sites

1. Partition $S$ into two disjoint subsets $S_1$ and $S_2$ of equal size
2. Construct $\text{Vor}_\rho(S_1)$ and $\text{Vor}_\rho(S_2)$ recursively
3. Merge the two Voronoi diagrams to obtain $\text{Vor}_\rho(S)$
The Merging Step

1. Overlay $\text{Vor}_\rho(S_1)$ and $\text{Vor}_\rho(S_2)$
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3. Label feature of the refined overlay with the sites nearest to it
The Merging Step

1. Overlay $\text{Vor}_\rho(S_1)$ and $\text{Vor}_\rho(S_2)$
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3. Label feature of the refined overlay with the sites nearest to it
4. Remove redundant features
Power Distance

The **power distance** between a circle and a point in the plane:

\[
\rho(x, d_c, r) = \|x - c\|^2 - r^2
\]
Power Distance

The power distance between a circle and a point in the plane:

\[ \rho(x, d_c, r) = ||x - c||^2 - r^2 \]

- Approximates the Euclidean distance function, e.g., inside the circle the distance is negative, outside — positive
- Bisector passes through the intersection points of two circles
Power diagram

groups.csail.mit.edu

www.jaist.ac.jp
Power diagram

Applications:
- Determine whether a point is inside the union of $n$ circles
- Find the boundary of the union of $n$ circles
- Classify $n$ circles into connected components
  - Mobile phones — can establish a line only if in the same connected component
Constructing the Power Diagram

\[ \rho(x, d_i) = ||x - c_i||^2 - r_i^2 \]

The Voronoi diagram of 36 points
Constructing the Power Diagram

\[ \rho(x, d_i) = \|x - c_i\|^2 - r_i^2 \]

\[ f_i(x) = x^2 - 2xc_i + c_i^2 - r_i^2 \]

\[ \pi_i : -2xc_i + c_i^2 - r_i^2 \]

The Voronoi diagram of 36 points
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The Voronoi diagram of 36 points

**Corollary**

The bisectors of the power diagram of circles in the plane are lines. We can compute the power diagram of circles by computing the lower envelope of planes. In fact, all diagrams whose bisectors are lines are power diagrams.
Multiplicatively-weighted Voronoi diagram

- Multiplicatively-weighted Voronoi diagram of sites $s_i = (p_i, w_i)$ is defined by:
  $$\rho(x, s_i) = w_i \| x - p_i \|^2$$
- Useful, for example, in modelling the growth of crystals
- The complexity of the diagram could be quadratic
Möbius diagram

- The Möbius diagram is a generalization of the multiplicatively-weighted Voronoi diagram and is defined by:
  \[ \rho(x, s_i) = w_i ||x - p_i||^2 + v_i \]
- The bisectors of the diagram are circles
- Every Voronoi diagram with circles as bisectors is Möbius diagram
Apollonius Diagrams

- The Apollonius diagram of disks $d_i = (p_i, r_i)$ is defined by the following distance function:

$$\rho(x, d_i) = \|x - p_i\| - r_i$$

- Negative distance inside the disks
- Bisectors are hyperbolic arcs
- Useful for . . .
An annulus is the bounded area between two concentric circles.

The width of an annulus is the difference between its radii $R$ and $r$.

**Goal**: given a set $S$ of objects (points, segments, etc.) find an annulus of minimum width containing the objects (MWA).
Tolerancing Metrology

- **Roundness** is the measure of sharpness of a particle’s edges and corners.
- In mechanical design there is a need to assess the roundness error of a manufactured object (to see it was manufactured correctly).

- 4 ANSI & ISO methods for round manufactured object assessment:
  - Minimum circumscribed circle (MCC)
  - Maximum inscribed circle (MIC)
  - Least square circle (LSC)
  - Minimum Zone circle (MZC)

www.npl.co.uk/server.php
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**Minimum-Width Annulus**
Facility Location

- Place a new cell tower given the location of clients’ cell phones
- Cell site has both desirable properties (supplement of service) and obnoxious properties (health)

cgm.cs.mcgill.ca/athens/cs507/Projects/2004/Emory-Merryman
Facility Location

- Place a new cell tower given the location of clients’ cell phones
- Cell site has both desirable properties (supplement of service) and obnoxious properties (health)
- A good location is the center of the minimum-width annulus
Related Work

Points:
- Rivlin [Riv79] and afterwards Smid and Janardan [SJ99] proved the theorem for points (following slides)
- Algorithms based on this theorem and Voronoi Diagrams introduced by Ebara [EFNN89] and by Roy and Zhang [RZ92]

Variants:
- Constrained MWA by de Berg et al. [dBBB⁺98] — useful enforcing different restrictions on roundness
- MWA of a polygon by Le and Lee [LL91]
  - Not identical to the MWA of segments
  - Controversy on the right algorithm for roundness
Related Work — cont.

- Special Sub-quadratic Algorithms:
  - Linear and $O(n \log n)$ algorithms for special cases with more information (Garcia-Lopez et al. ’98, Swanson et al. ’93, Devillers and Ramos ’02)
  - Best known randomized algorithm (with expected running time of $O(n^{3/2+\varepsilon})$) is achieved by solving a 3D width-like problem after lifting the points by $(x, y) \mapsto (x, y, x^2 + y^2)$
    - A circle $\mapsto$ a plane
    - 2 concentric circles $\mapsto$ 2 parallel planes

- Using coresets Chan [Cha06] presented an $(1 + \varepsilon)$-factor approximation algorithm
The Connection to Voronoi Diagrams

MWA does not always exist — This is possible in case of the width of the points is smaller than any width of any annulus (collinear points). 3 cases:
The Connection to Voronoi Diagrams

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*Inner circle* touches 3 points — center is a nearest Voronoi vertex
The Connection to Voronoi Diagrams

MWA does not always exist — This is possible in case of the width of the points is smaller than any width of any annulus (collinear points).

3 cases:

*Outer circle* touches 3 points — center is a *farthest Voronoi vertex*
The Connection to Voronoi Diagrams

MWA does not always exist — This is possible in case of the width of the points is smaller than any width of any annulus (collinear points).

3 cases:
Both inner and outer circles touches $\geq 2$ points — center is an intersection point between the diagrams (on edges of both diagrams)

With points if MWA exist only last case is possible
Proof

Observation

*Each of the circles touches at least one point*
Proof

Observation

*Each of the circles touches at least one point*

A point fixes an annulus
Lemma

At least one of the circles touches more than one point

Proof.

Suppose to contrary. Two cases:
- $A$ is on $OB$

\[ |PB| < |PA| + |AB| \]
\[ r = |AB| > |PB| - |PA| \]
Lemma

At least one of the circles touches more than one point

Proof.

Suppose to contrary. Two cases:

- $A$ is on $OB$
- $A$ is not on $OB$

\[
|OA| < |OP| + |PA|
\]

\[
r = |OB| - |OA| = |OP| + |PB| - |OA| > |PB| - |PA|
\]
Proof — cont.

Theorem

Each circle touches at least 2 points

Proof.

Suppose to contrary.
Case I — Out touches one point:
Out touches $Q$, In — $P_i$
- none of $P_i$ is on $OQ$

\[
|OP| > |OP_i| - |PP_i|\\
|OQ| - |OP_i| = |OP| + |PQ| - |OP_i| > |PQ| - |PP_i|
\]
Proof — cont.

Theorem

**Each circle touches at least 2 points**

Proof.

Suppose to contrary.

Case I — *Out* touches one point:

- *Out* touches $Q$, *In* — $P_i$
  - none of $P_i$ is on $OQ$
  - $P_1$ is on $OQ$

\[ |OP| + |PP_i| > |OP_i| = |OP| + |PP_1| \]

Annulus at $P$ is MWA, contradiction
Proof — cont.

Theorem

Each circle touches at least 2 points

Proof.

Suppose to contrary.

Case I — *Out* touches one point:

*Out* touches *Q*, *In* — *P*ₙ

- none of *P*ₙ is on *OQ*
- *P*₁ is on *OQ*

\[ |OP| + |PP_i| > |OP_i| = |OP| + |PP_1| \]

Annulus at *P* is MWA, contradiction
Proof — cont.

Theorem

Each circle touches at least 2 points

Proof.

Suppose to contrary.

Case I — Out touches one point:

Out touches Q, In — $P_i$

- none of $P_i$ is on $OQ$
- $P_1$ is on $OQ$

$|OP| + |PP_i| > |OP_i| = |OP| + |PP_1|$

Annulus at $P$ is MWA, contradiction

Case II (In touches 1 point) — similar
MWA of Disks in the Plane

Nearest Voronoi is replaced by the Apollonius diagram

\[ \delta(x, d_i) = ||x - c_i|| - r_i \]
MWA of Disks in the Plane

Nearest Voronoi is replaced by the Apollonius diagram. Farthest Apollonius diagram is not good in this case.

\[ \delta(x, d_i) = ||x - c_i|| - r_i \]

We need to consider the farthest point of the disk from a point.
MWA of Disks in the Plane

Nearest Voronoi is replaced by the Apollonius diagram

Farthest Apollonius diagram is not good in this case

Farthest-Point Farthest-Site VD replaces the farthest VD

$$\delta(x, d_i) = ||x - c_i|| - r_i$$

We need to consider the farthest point of the disk from a point

$$\delta(x, d_i) = ||x - c_i|| + r_i$$
MWA of Disks in the Plane

Nearest Voronoi is replaced by the Apollonius diagram

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Farthest-Point Farthest-Site VD replaces the farthest VD

\[
\delta(x, d_i) = ||x - c_i|| - r_i
\]

We need to consider the farthest point of the disk from a point

\[
\delta(x, d_i) = ||x - c_i|| + r_i
\]

Farthest-point farthest-site is a farthest-site Apollonius with negative radii and was easily produced using our framework.
MWA of Disks in the Plane

Cont.

VD construction via Lower Envelopes
MWA of Disks in the Plane
Cont.
Further information can be found at:


and at the TAU project page:

Further Reading I

Voronoi diagrams and arrangements.  

R. Klein.  
*Concrete and Abstract Voronoi Diagrams*, volume 400 of *Lecture Notes in Computer Science*.  

M. Meyerovitch.  
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Further Reading II

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Curved voronoi diagrams.
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A. Fabri and S. Pion.
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S. Pion, M. Teillaud, and C. P. Tsironiannis.
Geometric filtering of primitives on circular arcs.
Further Reading III

M. Sharir.
The clarkson-shor technique revisited and extended.
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The overlay of lower envelopes and its applications.

E. Berberich and M. Meyerovitch.
Computing envelopes of quadrics.
Further Reading IV


Further Reading V

E. Fogel, D. Halperin, L. Kettner, M. Teillaud, R. Wein and N. Wolpert
*Effective Computational Geometry for Curves and Surfaces.*
J.-D. Boissonnat and M. Teillaud (eds.)
Chapter 1: *Arrangements*

C. Burnikel, K. Mehlhorn, and S. Schirra.
How to compute the voronoi diagram of line segments: Theoretical and experimental results.
ISBN 3-540-58434-X.

Mark de Berg, Prosenjit Bose, David Bremner, Suneeta Ramaswami, and Gordon T. Wilfong.
Computing constrained minimum-width annuli of point sets.
Further Reading VI

Theodore J. Rivlin. 
Approximating by circles. 

Michiel H. M. Smid and Ravi Janardan. 
On the width and roundness of a set of points in the plane. 

Hiroyuki Ebara, Noriyuki Fukuyama, Hideo Nakano, and Yoshiro Nakanishi. 
Roundness algorithms using the Voronoi diagrams. 

Utpal Roy and Xuzeng Zhang. 
Establishment of a pair of concentric circles with the minimum radial separation for assessing roundness error. 
Further Reading VII

