Computational Geometry

Efi Fogel

Tel Aviv University

CGAL 2D Arrangements
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Outline

1. CGAL 2D Arrangements
   - Definitions & Complexity
   - Representation
   - Literature

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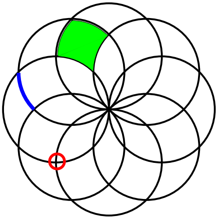
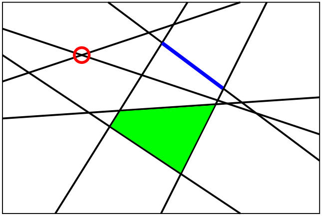
Two Dimensional Arrangements

Definition (Arrangement)

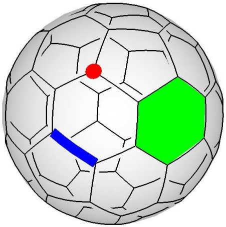
Given a collection $\mathcal{C}$ of curves on a surface, the arrangement $\mathcal{A}(\mathcal{C})$ is the partition of the surface into vertices, edges and faces induced by the curves of $\mathcal{C}$.



An arrangement of circles in the plane.



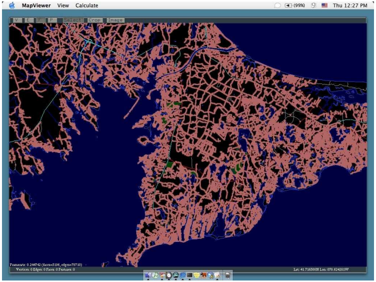
An arrangement of lines in the plane.

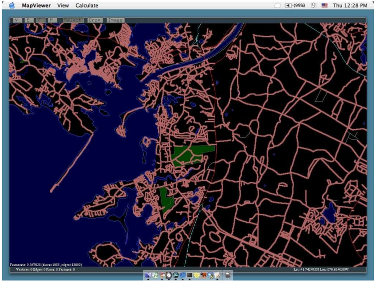


An arrangement of great-circle arcs on a sphere.

Arrangement Background

- Arrangements have numerous applications
  - robot motion planning, computer vision, GIS, optimization, computational molecular biology





A planar map of the Boston area showing the top of the arm of cape cod.

Raw data comes from the US Census 2000 TIGER/line data files

Arrangement 2D Complexity

Definition (Well Behaved Curves)

Curves in a set $\mathcal{C}$ are well behaved, if each pair of curves in $\mathcal{C}$ intersect at most some constant number of times.

Theorem (Arrangement in $\mathbb{R}^2$)

The maximum combinatorial complexity of an arrangement of n well-behaved curves in the plane is $\Theta(n^2)$.

The complexity of arrangements induced by n non-parallel lines is $\Omega(n^2)$.

Planar Maps

Definition (Planar Graph)

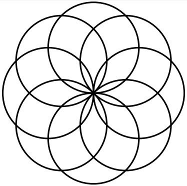
A planar graph is a graph that can be embedded in the plane.

Definition (Planar Map)

A planar map is the embedding of a planar graph in the plane. It is a subdivision of the plane into vertices, (bounded) edges, and faces.

Theorem (Euler Formula)

Let v, e, and f be the number of vertices, edges, and faces (including the unbounded face) of a planar map, then $v - e + f = 2$.



vertices — 25

edges — 56

faces — 33

Surface Maps

Planar maps generalize to surfaces!

Definition (genus)

A topologically invariant property of a surface defined as the largest number of nonintersecting simple closed curves that can be drawn on the surface without separating it.

Theorem (Euler Formula)

Let v, e, and f be the number of vertices, edges, and faces of a map embedded on a surface with genus g, then $v - e + f = 2 - 2g$.

If each face is incident to at least 3 edges $\implies 3f \leq 2e$

$$3v - 3e + 3f = 6 - 6g \leq 3v - 3e + 2e$$

$$e \leq 3v - 6 + 6g$$

In a planar triangulation $e = 3v - 6,\ f = 2v - 4$

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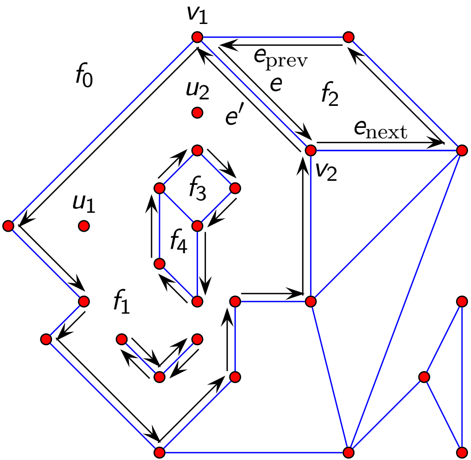
The Doubly-Connected Edge List

- One of a family of combinatorial data-structures called the *halfedge data-structures*.
- Represents each edge using a pair of directed *halfedges*.
- Maintains incidence relations among cells of 0 (vertex), 1 (edge), and 2 (face) dimensions.



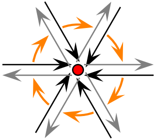
- The target vertex of a halfedge and the halefedge are incident to each other.
- The source and target vertices of a halfedge are adjacent.

The Doubly-Connected Edge List Components

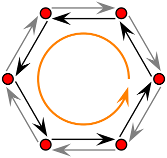
- Vertex
  - An incident halfedge pointing at the vertex.
- Halfedge
  - The opposite halfedge.
  - The previous halfedge in the component boundary.
  - The next halfedge in the component boundary.
  - The target vertex of the halfedge.
  - The incident face.
- Face
  - An incident halfedge on the outer CCB.
  - An incident halfedge on each inner CCB.
- Connected component of the boundary (CCB)
  - The circular chains of halfedges around faces.

Arrangement Representation

- The halfedges incident to a vertex form a circular list.
- The halfedges are sorted in clockwise order around the vertex.

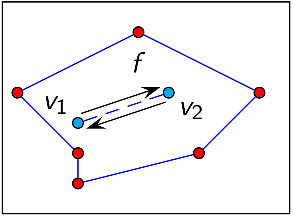


- The halfedges around faces form circular chains.
- All halfedges of a chain are incident to the same face.
- The halfedges are sorted in counterclockwise order along the boundary.

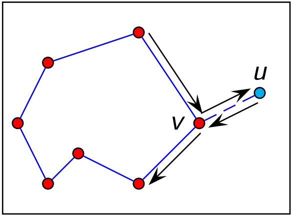


- Geometric interpretation is added by classes built on top of the halfedge data-structure.

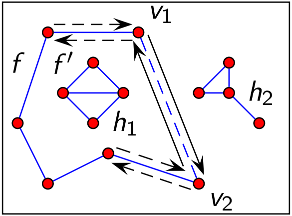
Modifying the Arrangement



Inserting a curve that induces a new hole inside the face f, `arr.insert_in_face_interior(c,f)`.



Inserting a curve from an existing vertex u that corresponds to one of its endpoints, `insert_from_left_vertex(c,v)`, `insert_from_right_vertex(c,v)`.



Inserting an x-monotone curve, the endpoints of which correspond to existing vertices v_1 and v_2, `insert_at_vertices(c,v1,v2)`.

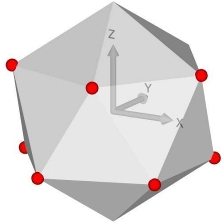
- The new pair of halfedges close a new face f'.
- The hole h_1, which belonged to f before the insertion, becomes a hole in this new face.

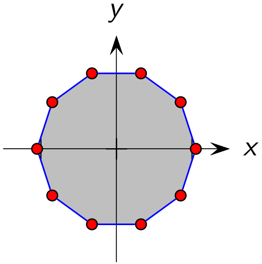
Application: Obtaining Silhouettes of Polytopes

Application

Given a convex polytope P obtain the outline of the shadow of P cast on the xy-plane, where the scene is illuminated by a light source at infinity directed along the negative z-axis.

- The silhouette is represented as an arrangement with two faces:
  - an unbounded face and
  - a single hole inside the unbounded face.





An icosahedron and its silhouette.

Application: Obtaining Silhouettes of Polytopes: Insertion

- Insert an edge into the arrangement only once to avoid overlaps.
  - Maintain a set of handles to polytope edges the projection of which have already been inserted into the arrangement.
  - Implemented with the std :: set data-structure.
    - ★ Requires the provision of a model of the *StrictWeakOrdering*.
    - ★ A functor that compares handles:

```
struct Less_than_handle {
  template <typename Type>
  bool operator()(Type s1, Type s2) const { return (&(*s1) < &(*s2)); }
};
```

```
std::set<Polyhedron_halfedge_const_handle, Less_than_handle>}
```

- Determine the appropriate insertion routines.
  - Maintain a map that maps polyhedron vertices to corresponding arrangement vertices.
  - Implemented with the std :: map data-structure.

```
std::map<typename Polyhedron::Vertex_const_handle,
         typename Arrangement::Vertex_handle, Less_than_handle>
```

Application: Obtaining Silhouettes of Polytopes: Construction

Obtain the arrangement $\mathcal{A}$ that represents the silhouette of a Convex Polytope P
1. Construct the input convex polytope P.
2. Compute the normals to all facets of P.
3. **for each** facet f of P
4.   **if** f is facing upwards (has a positive z component)
5.   **for each** edge e on the boundary of f
6.   **if** the projection of e hasn't been inserted yet into $\mathcal{A}$
7.   Insert the projection of e into $\mathcal{A}$.

Computes the normal to a facet.

```
struct Normal_equation {
  template <typename Facet> typename Facet::Plane_3 operator()(Facet & f) {
    typename Facet::Halfedge_handle h = f.halfedge();
    return CGAL::cross_product(h->next()->vertex()->point() -
                               h->vertex()->point(),
                               h->next()->next()->vertex()->point() -
                               h->next()->vertex()->point());
  }
};
```

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