A triangulation of set $S$ of points in the plane is a partition of the convex hull of the set into triangles whose vertices are the points, and do not contain other points. (Why is there always a triangulation?!) An alternative definition: A maximal collection of line-segments inside $\text{CH}(S)$ whose endpoints are points of $S$. (These segments form the triangles.) There are an exponential number of triangulations of a point set. Best known bound: $O(43^n)$ [Sharir and Welzl, 2006].
Motivation

- Assume a height value is associated with each point.
- A triangulation of the points defines a piecewise-linear surface of triangular patches.

Piecewise-Linear Interpolation

- The height of a point $P$ inside a triangle is determined by the height of the triangle vertices, and the location of $P$.
- The result depends on the triangulation.
Barycentric Coordinates

Any point inside a triangle can be expressed uniquely as a convex combination of the triangle vertices:

\[ p = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \]

\[ \alpha_i = \frac{A_i}{A_1 + A_2 + A_3} \quad \text{for} \quad 1 \leq i \leq 3 \]

\[ \alpha_i \geq 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 1 \]

(back to) Piecewise-Linear Interpolation

\[ \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = x_p \]
\[ \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = y_p \]
\[ \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 - z_p = 0 \]
\[ \alpha_1 + \alpha_2 + \alpha_3 = 1 \]
An $O(n^3)$-Time Triangulation Algorithm

- Repeat
  - Select two sites.
  - If the edge connecting them does not intersect previously kept edges, keep it.
- Until all faces are triangles.
- Question: Why $O(n^3)$ time?
- Question: Why is the algorithm guaranteed to stop before running out of edges?
- Answer: Because every nontriangular face has a diagonal that was not processed yet. (Why?!)
Number of Triangles

- The number of triangles $t$ in a triangulation of $n$ points depends on the number of vertices $h$ on the convex hull: $t = (h-2) + 2(n-h) = 2n-h-2$.

![Diagram showing triangulation examples]

$n = 8$

$h = 6 \rightarrow t = 8$

$h = 5 \rightarrow t = 9$

Quality Triangulations

- Consider a triangulation $T$.
- Let $\alpha(T) = (\alpha_1, \alpha_2, \ldots, \alpha_3)$ be the vector of angles in the triangulation $T$ sorted in increasing order.
- A triangulation $T_1$ is “better” than $T_2$ if $\alpha(T_1) > \alpha(T_2)$ (compared lexicographically).
- The Delaunay triangulation is the “best” (avoiding, as much as possible, long skinny triangles).

Good:  

Bad:
Thales’s Theorem

- **Theorem:**
  Let $C$ be a circle, and $\ell$ a line intersecting $C$ at the points $a$ and $b$. Let $p, q, r,$ and $s$ be points lying on the same side of $\ell$, where $p$ and $q$ are on $C$, $r$ inside $C$, and $s$ outside $C$. Then:
  \[
  \angle arb > \angle apb = \angle aqb > \angle ash
  \]

- **Proof omitted.**
  (Thales proved the theorem directly; one can deduce it from the sine theorem.)

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Improving a Triangulation

- **In any convex quadrangle, an edge flip is possible.**
  (Why? Why isn’t it possible in a concave triangle?)

- **Claim:** If this flip improves the triangulation locally, it also improves the global triangulation. (Why?)

- **If an edge flip improves the triangulation (locally and hence globally), the original edge is called illegal.**
Illegal Edges

- **Lemma:** An edge $pq$ is illegal iff any of its opposite vertices is inside the circle defined by the other three vertices.
- **Proof:** By Thales’s theorem.
- Moreover, a convex quadrangle in general position has exactly one legal diagonal.
- **Theorem:** A Delaunay triangulation does not contain illegal edges. (Otherwise it can be improved locally.)
- **Corollary:** A triangle is Delaunay iff the circle through its vertices is empty of other sites.
- **Observation:** The Delaunay triangulation is not unique if more than three sites are cocircular.

An $\Theta(n^4)$-Time Delaunay Triangulation

- For all triples of sites:
  - If the circle through the triple of sites does not contain any other sites, keep the triangle whose vertices are the triple.

- **Complexity:** $\Theta(n^3)$ triples, $\Theta(n)$ work on each triple; Total: $\Theta(n^4)$ time.

(Space complexity: $\Theta(n)$.)
The In-Circle Test

**Theorem:** If $a, b, c, d$ form a CCW convex polygon, then $d$ lies in the circle determined by $a$, $b$, and $c$ iff:

\[
\det \begin{pmatrix}
a_x & a_y & a_x^2 + a_y^2 & 1 \\
b_x & b_y & b_x^2 + b_y^2 & 1 \\
c_x & c_y & c_x^2 + c_y^2 & 1 \\
d_x & d_y & d_x^2 + d_y^2 & 1
\end{pmatrix} > 0
\]

**Proof:**
We prove that equality holds if the points are cocircular.

There exists a center $q$ and radius $r$ such that:
\[
(a_x - q_x)^2 + (a_y - q_y)^2 = r^2
\]

Similarly for $b$, $c$, $d$.

In vector notation:
\[
\begin{pmatrix}
a_x \\
b_x \\
c_x \\
d_x
\end{pmatrix} \begin{pmatrix}
a_x \\
b_x \\
c_x \\
d_x
\end{pmatrix} - 2q_i \begin{pmatrix}
a_y \\
b_y \\
c_y \\
d_y
\end{pmatrix} + (q_i^2 + q_i^2 - r^2) = 0
\]

So these four vectors are linearly dependent, and hence their determinant vanishes.

**Corollary:** $d \in o(a, b, c)$ iff $b \in o(a, c, d)$ iff $c \in o(b, a, d)$ iff $a \in o(b, c, d)$.

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Naive Delaunay Algorithm

- Start with an arbitrary triangulation.
- Flip any illegal edge until no more exist.
Naive Delaunay Algorithm (cont.)

- Question: Why does the algorithm terminate?
  - Answer: Because every flip increases the vector angle, and there are finitely-many such vectors.
  - However, this algorithm is in practice very slow.

- Question: Why does the algorithm converge to the optimum triangulation?
  - Answer: Because there are no local maxima (proof deferred).

Delaunay Triangulation by Duality

- Draw the Delaunay graph (the dual graph of the Voronoi diagram) by connecting each pair of neighboring sites in the Voronoi diagram.
- If no four points are cocircular, then the Delaunay graph is triangulated.
- General position assumption: There are no four cocircular points.
- We need to prove:
  - Correctness of this duality. That is, that drawing the Delaunay graph with straight segments does not cause any segment intersection.
  - That this triangulation indeed maximizes the angle vector (and, hence, it is the Delaunay triangulation).
- **Corollary:** The Delaunay triangulation (DT) of \( n \) points can be computed in \( O(n \log n) \) time.
Proof of Planarity of Delaunay Triangulation

- Let \( S \) be a set of sites, and let \( DT(S) \) be the dual graph of \( VD(S) \).
- Let \( p_ip_j \) be an edge of \( DT(S) \).
  It is so because cells of \( p_i \) and \( p_j \) in \( VD(S) \) are neighbors in \( VD(S) \).
  Hence, there exists an \textbf{empty} circle passing through \( p_i, p_j \) and whose center \( o_{ij} \) is on their bisector (the edge of \( VD(S) \) separating between the cells of \( p_i \) and \( p_j \)).

- Assume for contradiction that \( p_ip_j \) intersects another edge \( p_kp_l \) in \( DT(S) \).

- Observe the possible interactions between the triangles \( \Delta o_{ij}p_ip_j \) and \( \Delta o_{kl}p_kp_l \)… (next slide)

Planarity Proof (cont.)

- Case A (one triangle contains a yellow vertex of the other triangle):
  Impossible, since the circumscribing circle of the first triangle is empty, hence also the triangle.

- Case B (no triangle contains a vertex of the other triangle):
  Cannot avoid an intersection of a pair of white edges, which is impossible, because the white edges are fully contained in \textbf{disjoint} Voronoi cells.

- Case C (one triangle contains a green vertex of the other triangle) is possible.
  Question: Why isn’t it a contradiction?
Delaunay Triangulation: Main Property

- **Theorem:**
  Let $S$ be a set of points in the plane. Then,
  
  (i) $p_i, p_j, p_k \in S$ are vertices of a triangle (face) of $\text{DT}(S)$ if and only if the circle passing through $p_i, p_j, p_k$ is empty.
  
  (ii) $p_i, p_j$ (for $p_i, p_j \in S$) is an edge of $\text{DT}(S)$ if and only if there exists an empty circle passing through $p_i, p_j$.

- **Proof:** Dualize the Voronoi-diagram theorem.

- **Corollary:**
  A triangulation $T(S)$ is $\text{DT}(S)$ if and only if every circumscribing circle of a triangle $\Delta \in T(S)$ is empty.

Wrapping Up

- **Theorem:**
  Let $S$ be a set of points in the plane, and let $T(S)$ be a triangulation of $S$. Then,
  
  $T(S) = \text{DT}(S)$ if and only if $T(S)$ is legal.

- **Proof:** Follows from the definitions of a legal edge and triangulation. (Exercise!)

- **Corollary:** $\text{DT}(S)$ maximizes the vector angle.

  Since $\text{DT}(S)$ is unique, there is only one legal triangulation, and thus, there are no local maxima in the edge-flip algorithm. That is, it converges to $\text{DT}(S)$. 
An $O(n \log n)$-Time Delaunay Algorithm

A **Randomized** incremental algorithm:

- Form a bounding triangle $\Delta_0$ enclosing all points.
- Add the points one after another in random order and update the triangulation.
- If the point is inside an existing triangle:
  - Connect the point to the triangle vertices.
  - Check if a flip can be performed on any of the three triangle edges. If so, flip the edge and check recursively the neighboring edges (opposite to the new point).
- If the site is on an existing edge:
  - Replace the edge with four new edges.
  - Check if a flip can be performed on any of the opposite edges. If so, flip the edge and check recursively the neighboring edges (opposite to the new point).

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Flipping Edges

- A new point $p_i$ was added, causing the creation of the edges $p_ip_r$ and $p_ip_j$.
- The legality of the edge $p_ip_j$ (with opposite vertex) $p_k$ is checked.
- If $p_ip_j$ is illegal, perform a flip, and recursively check edges $p_ip_k$ and $p_ip_k$, the new edges opposite to $p_r$.
- Notice that the recursive call for $p_ip_k$ cannot eliminate the edge $p_rp_k$.
- **Note**: All edge flips replace edges opposite to the new vertex by edges adjacent to it!

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Flipping Edges: Example

Theorem: The expected number of triangles created in the course of the algorithm (some of which also disappear) is at most $9n+1$.

Proof:
During the insertion of point $p_i$, $k_i$ new edges are created: 3 new initial edges, and $k_i-3$ due to flips. Hence, the number of new triangles is at most $3 + 2(k_i-3) = 2k_i-3$. (A point on an edge results in $2k_i-4$ triangles.)

What is the expected value of $k_i$?
Number of Triangles (cont.)

- Recall that the Voronoi diagram has at most $3N - 6$ edges, where $N$ is the number of vertices.
- The number of edges in a graph and its dual are identical.
- Taking into account the initial triangle $\Delta_0$, after inserting $i$ points, there are at most $3(i+3) - 6 = 3i + 3$ edges. Three of them belong to $\Delta_0$, so we are left with at most $3i$ internal edges that are adjacent to the input points.

Number of Triangles (cont.)

- The sum of all vertex degrees is thus at most $2 \cdot 3i = 6i$.
- On the average, the degree of each vertex is only $6$! But this is exactly the number of new edges!
- Hence, the expected number of triangles created in the $i$th step is at most $E(2k_i - 3) = 2E(k_i) - 3 = 9$.
- Therefore, the expected number of triangles created (and possibly destroyed) for $n$ points is $9n + 1$. (One initial bounding triangle plus 9 triangles on average per point.)
Algorithm Complexity

- Point location for every point: \( O(\log n) \) time (not shown).
- Flips: \( \Theta(n) \) expected time in total (for all steps).
- Total expected time: \( O(n \log n) \).
- Space: \( \Theta(n) \).

Relatives of the Delaunay Triangulation

- **Euclidean Minimum Spanning Tree (EMST):**
  A tree of minimum length connecting all the sites.
- **Relative Neighborhood Graph (RNG):**
  Two sites \( p, q \) are connected if
  \[
  d(p, q) \leq \min_{r \in P, r \neq p, q} \max(d(p, r), d(q, r))
  \]
- **Gabriel Graph (GG):**
  Two sites \( p, q \) are connected if the circle whose diameter is \( pq \) is empty of other sites.
- **Theorem:** \( \text{EMST} \subseteq \text{RNG} \subseteq \text{GG} \subseteq \text{DT} \).
Delaunay Triangulation and Convex Hulls

Project the 2D point set onto the 3D paraboloid
Compute the 3D lower convex hull
Project the 3D facets Back to the plane.

The 2D triangulation is Delaunay!

Proof of Lift-Up

- The intersection of a plane with the paraboloid is an ellipse whose projection to the plane is a circle.
- \( s \) lies within the circumcircle of \( p,q,r \) iff \( s' \) lies on the lower side of the plane passing through \( p',q',r' \).

- \( p, q, r \in S \) form a Delaunay triangle iff \( p', q', r' \) form a face of the convex hull of \( S' \).
Given a set $S$ of points in the plane, associate with each point $p=(a,b) \in S$ the plane $z = 2ax + 2by - (a^2 + b^2)$, which is tangent to the paraboloid at $p'$, the vertical projection of $p$ onto the paraboloid.

VD($S$) is the vertical projection onto the $XY$ plane of the boundary of the convex polyhedron that is the intersection of the halfspaces above these planes.