

Programming Assignment (optional)

due: May 7th, 2012

The exercise given in this assignment is a variant of exercise 5.7 in [3, Chapter 5]. Please submit the solution to this assignment directly to Efi Fogel (efifogel@gmail.com). Further details will be given in the course website.



Fig. 1: A sunflower capitulum with a parastichy pair of 55 spirals winding clockwise and 34 spirals winding counterclockwise.

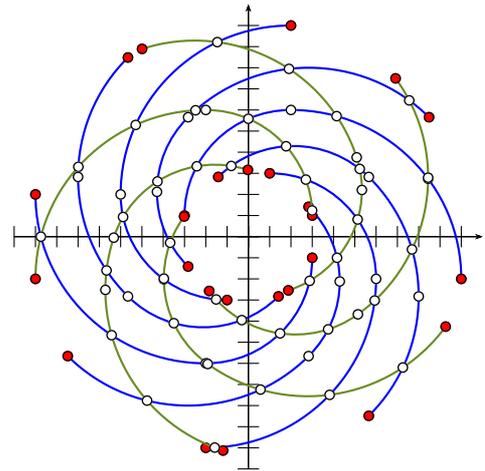
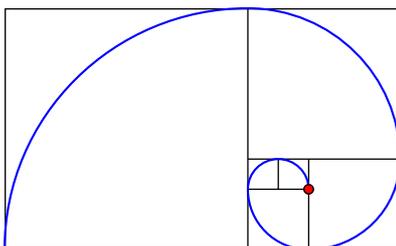


Fig. 2: A parastichy pair of Fibonacci spirals. The clockwise-winding set (blue) and counterclockwise-winding set (green) consists of eight and five spirals, respectively. Each spiral arc in both sets comprises two circular arcs, the fifth and the sixth circular arcs in the underlying complete Fibonacci spiral.

Exercise: Pattern formation in organisms is a spectacular phenomenon observed in nature. One family of such patterns includes the regular arrangements of outer plant organs such as leaves, blossoms or seeds along branches or in a bud. In the botanical literature these patterns are called phyllotactic patterns. The most common phyllotactic pattern is the spiral pattern, where a single primordium is inserted at each node. A classical example of a spiral pattern is the arrangement of florets (or their mature form, seeds) in the capitula (commonly referred to as heads) of sunflowers. As the name suggests, it is possible to trace spiral curves in the pattern. These curves are called parastichies. The parastichies winding in the same direction around the stem with the same pitch constitute one family of parastichies, and two easily observed sets of parastichies winding in opposite directions are called a parastichy pair. The numbers of parastichies in the two sets of a parastichy pair are consecutive Fibonacci numbers. The number of parastichies decreases towards the center; see Figure 1. For more details see, e.g., [2, 4].



In this exercise you are asked to compute all intersections of a parastichy pair. Parastichies have been shown to have the paths of logarithmic spirals. However, the curves you are asked to handle are Fibonacci spirals, which well approximate logarithmic spirals on the one hand and are easier to handle on the other. A Fibonacci spiral is a piecewise circular curve comprising circular arcs (quarter circles) connecting the opposite corners of squares in the tiling with squares whose sides are successive Fibonacci numbers in length. The Fibonacci counterclockwise spiral arc depicted in the figure to the left uses squares of sizes 1, 1, 2, 3, 5, and 8. The small red circle indicates the spiral source-point. Reflecting the counterclockwise-winding spiral arc about any line containing the source-point results with a clockwise-winding spiral arc.

1. Develop a function that constructs a single counterclockwise-winding Fibonacci spiral arc. It accepts three positive integers, k , m , and s , that define the resulting Fibonacci spiral arc as follows: Let c_1, c_2, \dots denote the circular arcs that sequentially compose the (unscaled) Fibonacci spiral. The desired Fibonacci spiral arc comprises $c'_k, c'_{k+1}, \dots, c'_{k+m-1}$, where c'_i is the circular arc c_i whose radius is scaled by s . The source point is the bottom right corner of the square bounding c_1 . For example, the spiral arc depicted at the bottom of the previous page is defined by 1, 6, 1.
2. Develop a program that computes all intersection points of Fibonacci spiral arcs in a given parastichy pair. The program accepts eight arguments, $k_1, m_1, s_1, k_2, m_2, s_2, n$, and ϵ as input. k_1, m_1, s_1 are three integers that define the clockwise-winding spiral that is the prototype of spirals in the first set of the parastichy pair. k_2, m_2, s_2 are three integers that define the counterclockwise-winding spiral that is the prototype of spirals in the second set. The next input, n , is an integer that indirectly specifies the number of spirals in the two sets as follows: Let f_i denote the i th Fibonacci number. Then, f_n and f_{n+1} are the numbers of spirals in the two sets, respectively. For example, the prototypes of the clockwise-winding and the counterclockwise-winding sets of spirals depicted in Figure 2 are both defined by the triplet $\langle 5, 2, 1 \rangle$. In order to obtain the parastichy pair depicted in this figure, n must be set to 5. The last argument, ϵ , indicates an error bound. It is given as two integers, a and b , such that $\epsilon = a/b$.

The program is given a name of an input text file on the command line. The input file contains all arguments. The program parses the file and extracts the arguments. Then, the program constructs two prototype spiral arcs, say a_1 and a_2 , winding in opposite orientations such that the source points of the underlying complete spirals of which coincide at the origin, and the clockwise-winding arc is obtained by reflecting the corresponding counterclockwise-winding arc about the line $x + y = 0$. Next, it rotates a_1 by the angles $\alpha_1, \dots, \alpha_{f_n}$ about the origin to construct one set of f_n spirals, where α_i approximates the angle $i \frac{2\pi}{f_n}$. Then, it rotates a_2 by the angles $\beta_1, \dots, \beta_{f_{n+1}}$ about the origin to construct the second set of f_{n+1} spirals, where β_i approximates the angle $i \frac{2\pi}{f_{n+1}}$. The approximations must satisfy the following conditions: $|\sin \alpha_i - \sin \alpha'_i| \leq \epsilon$, $|\cos \alpha_i - \cos \alpha'_i| \leq \epsilon$, $|\sin \beta_i - \sin \beta'_i| \leq \epsilon$, and $|\cos \beta_i - \cos \beta'_i| \leq \epsilon$, where α'_i and β'_i are $i \frac{2\pi}{f_n}$ and $i \frac{2\pi}{f_{n+1}}$, respectively, calculated using double precision floating-point arithmetic, $\pi = 4 \arctan(1)$ is computed using `std::atan(1)`, and `sin(.)` and `cos(.)` are computed using `std::sin(.)` and `std::cos(.)`, respectively.

In order to stay in the realm of rational numbers, you may use the free function `rational_rotation_approximation()` provided by CGAL to obtain the desired $\sin \alpha_i$ and $\cos \alpha_i$ for $i = 1, \dots, f_n$, and $\sin \beta_i$ and $\cos \beta_i$ for $i = 1, \dots, f_{n+1}$, or better yet, the approximate rotation constructor of the `Aff_transformation_2` construct provided by any two-dimensional kernel of CGAL to obtain the desired rotation directly. The implementations of these procedures are based on a method presented by Canny, Donald, and Ressler [1].

The program computes the intersection points of all the Fibonacci spirals in the constructed parastichy pair using the two methods listed below and compares their performance. The output points should include end-points and points that split the spirals into x -monotone sub-curves. However, duplications should be avoided.

- (a) Every pair of spiral arcs is processed to detect intersections between the two arcs of the pair.
- (b) All spiral arcs are processed at once to compute all intersections using the free function `compute_intersection_points()`, which exploits the plane-sweep paradigm.

You may assume that the constructed clockwise-winding spiral arcs are pairwise disjoint except perhaps at the origin; the same holds for the counterclockwise-winding spiral arcs.

The output should consist of three numbers: The number of computed points followed by the time in seconds it took to compute the points using the two methods. Additional output requirements will be posted in the course Web page.

Optional: Add visualization capabilities to render the parastichy pair and the computed intersection points into a graphical window. It is advised that you use QT.

References

- [1] J. Canny, B. Donald, and E. K. Ressler. A rational rotation method for robust geometric algorithms. In *Proceedings of the 8th Annual ACM Symposium on Computational Geometry (SoCG)*, pages 251–260. Association for Computing Machinery (ACM) Press, 1992.
- [2] O. Deussen and B. Lintermann. *Digital Design of Nature: Computer Generated Plants and Organics*. Springer, Berlin, Heidelberg, Germany, 2005.
- [3] Efi Fogel, Ron Wein, and Dan Halperin. *CGAL Arrangements and Their Applications, A Step-by-Step Guide*. Springer, 2012.
- [4] R. V. Jean. *Phyllotaxis: A Systemic Study in Plant Morphogenesis*. Cambridge University Press, Cambridge, UK, 1994. Cambridge Books Online.