Logarithmic-Time Point Location in General Two-Dimensional Subdivisions

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Tel Aviv University, May 2014

Joint work with Michael Hemmer and Dan Halperin

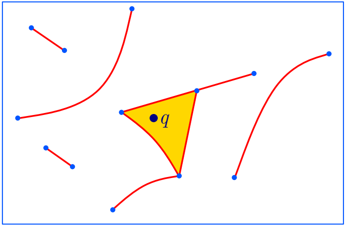
Planar Point Location - Definition

- Let S be a planar subdivision consisting of faces, edges, and vertices

The Planar Point Location Problem

Input: Query point q

Output: The feature of S containing q



n - the number of subdivision edges

Outline

- Trapezoidal-map RIC point-location variants
- Depth vs. maximum query path length
- An efficient construction algorithm for static settings
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Two Variants of the Trapezoidal Map RIC Point Location

- **Basic algorithm** [Mulmuley '90, Seidel '91]
  - Expected $O(\log n)$ query time
  - Expected $O(n)$ size
  - Expected $O(n \log n)$ preprocessing time
- **Guaranteed variant** [de Berg et al. '00]
  - Guaranteed $O(\log n)$ query time
  - Guaranteed $O(n)$ size
  - Expected $O(n \log^2 n)$ preprocessing time (?)

The Basic RIC Point Location Algorithm

Description: Builds the trapezoidal-map using a randomized incremental construction and maintains an auxiliary search-structure (DAG)

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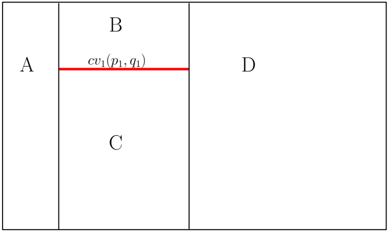

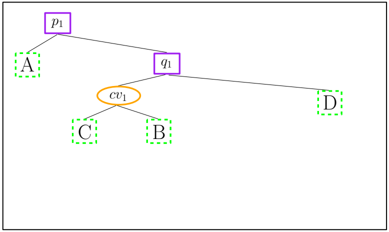

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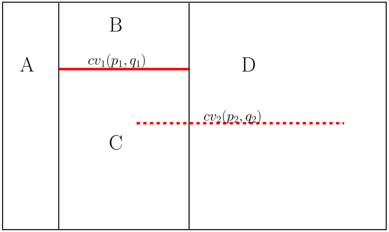

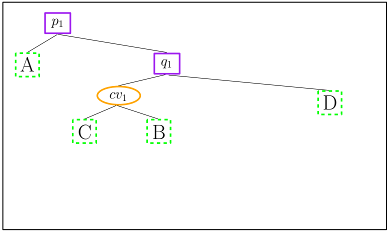

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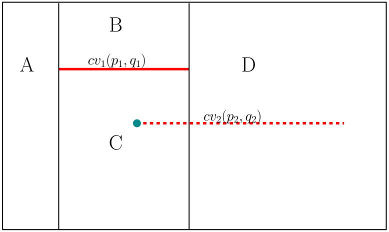

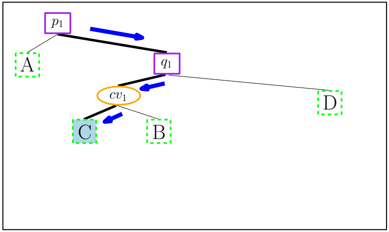

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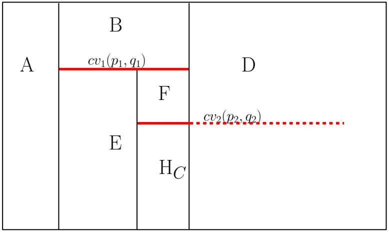

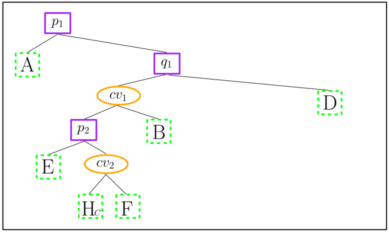

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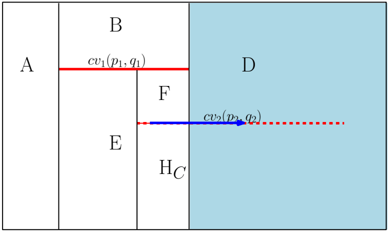

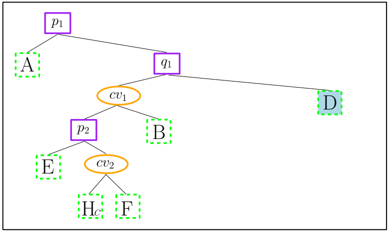

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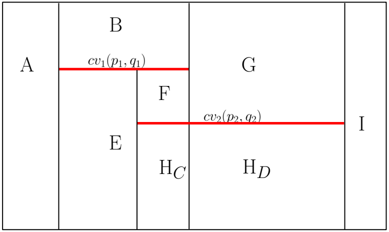

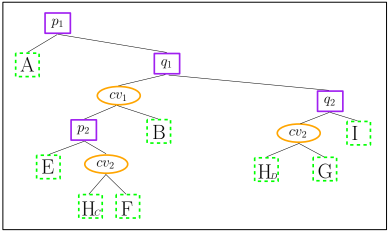

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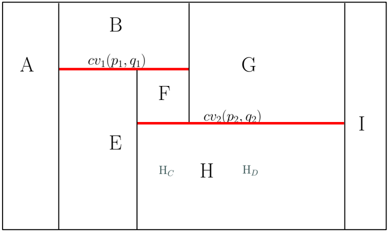

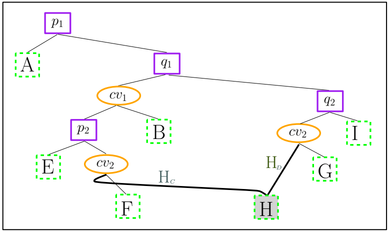

[Mulmuley '90, Seidel'91]

Basic Algorithm [Mulmuley '90, Seidel '91] - Complexity

- Expected $O(\log n)$ query time
- Expected $O(n)$ size
- Expected $O(n \log n)$ preprocessing time

Guaranteed $O(\log n)$ Query Time and $O(n)$ Size

- $\mathcal{S}$ - the size of the DAG
- $\mathcal{L}$ - the length of the longest query path

[de Berg et al.]

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The main idea:

- Construct the DAG using the basic algorithm with some random insertion order
  - Verify $\mathcal{S}$ on the fly ($\mathcal{S}$ can be accessed in $O(1)$ time)
  - Abort and rebuild if $\mathcal{S} \geq c_1 n$
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  - Abort and rebuild if $\mathcal{S} \geq c_1 n$
- Verify that $\mathcal{L} \leq c_2 \log n$, rebuild otherwise
- Only a constant number of rebuilds is expected
  - The probability that $\mathcal{L}$ is bad is very small (Lemma 2)
  - The probability that $\mathcal{S}$ is bad is very small (Lemma 3)

[de Berg et al.]

Guaranteed $O(\log n)$ Query Time and $O(n)$ Size

- $f(n)$ - Time to verify that $\mathcal{L}$ is logarithmic on a DAG of n curves
- Overall expected time for construction: $O(n \log n + f(n))$
- It is unclear how to efficiently verify $\mathcal{L}$:
  - Claim that the expected verification time is $O(n \log^2 n)$
  - No concrete proof is given

[de Berg et al.]

The Probabilities for "Bad" $\mathcal{L}$ or $\mathcal{S}$ are Small

Lemma 1 (Prob. that a given search path is bad is small)

Given a set S of n non-crossing line segments, a query point q, and a parameter $\lambda > 0$, the probability that the search path for q in the DAG has more than $3\lambda \log(n+1)$ nodes is at most $1/(n+1)^{\lambda \log 1.25 - 1}$

Proof: using a tail estimate (Appendix)

The Probabilities for "Bad" $\mathcal{L}$ or $\mathcal{S}$ are Small

Lemma 2 (Prob. that $\mathcal{L}$ is bad is small)

Given a set S of n non-crossing line segments, and a parameter $\lambda > 0$, the probability that the maximum length of a search path in the DAG is more than $3\lambda \log(n+1)$ is at most $2/(n+1)^{\lambda \log 1.25 - 3}$

Proof sketch:

- Extend vertical walls at each endpoint- defining at most $2(n+1)^2$ regions
- Consider the search paths of representative points of these regions
- By Lemma 1 we get the required result

The Probabilities for "Bad" $\mathcal{L}$ or $\mathcal{S}$ are Small

Lemma 3 (Prob. that $\mathcal{S}$ is bad is small)

Given a set S of n non-crossing x-monotone curves, and a parameter $\rho \geq 1$, the probability that the size $\mathcal{S}$ of the DAG is greater than $15\rho n$ is at most $1/\rho$

(Proof in the Appendix)

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- $\mathcal{D}$ represents the length of the longest DAG path

The Modified Algorithm Using $\mathcal{D}$

The modified algorithm:

- Observe $\mathcal{S}$ and $\mathcal{D}$ during construction
- Abort and rebuild structure if one of the following occurs:
  - $\mathcal{S} \geq c_1 n$
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- $\mathcal{D}$ is not $\mathcal{L}$
  - Can we still expect a constant number of rebuilds?

The Difference between $\mathcal{D}$ and $\mathcal{L}$

- Reminder: $\mathcal{D}$ represents the length of the longest DAG path
- Some DAG paths are not search paths
  - $\mathcal{D}$ is an upper bound on $\mathcal{L}$
  - $\mathcal{D}$ may be significantly larger than $\mathcal{L}$

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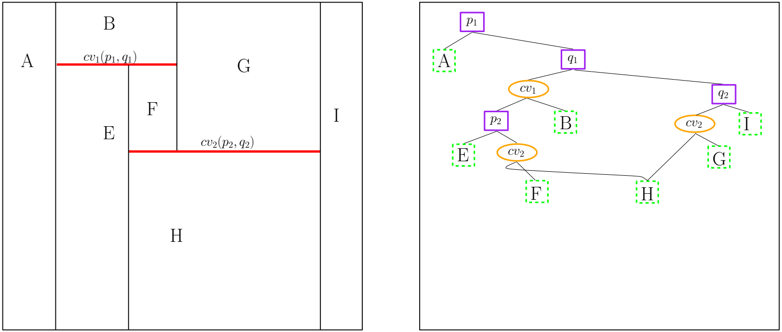

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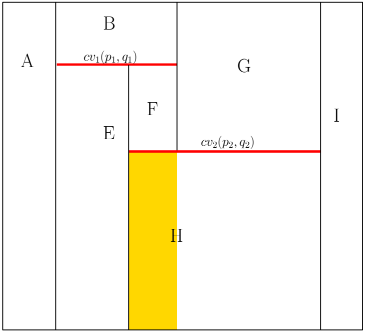

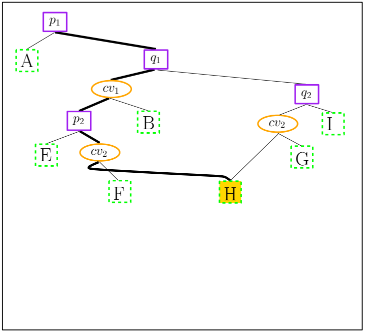

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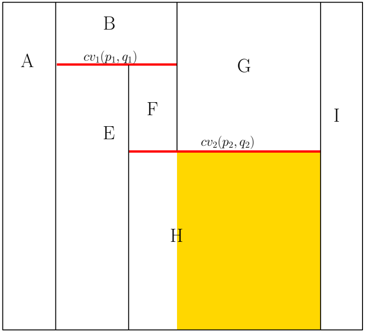

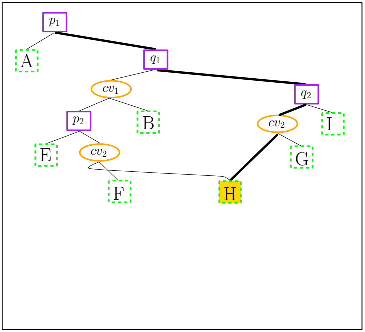

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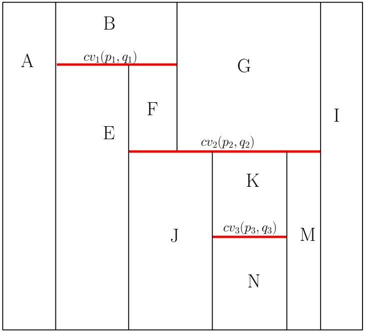

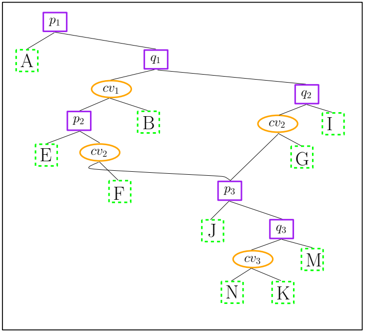

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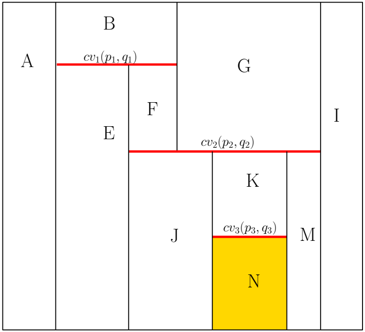

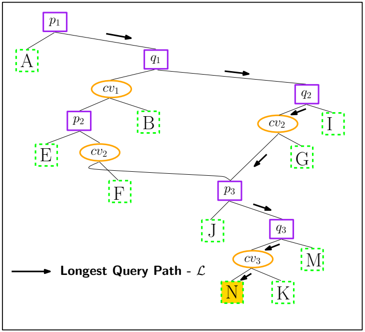

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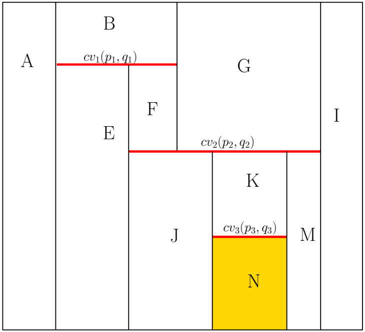

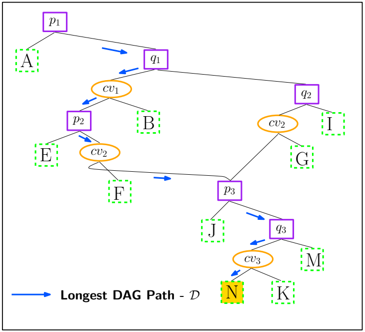

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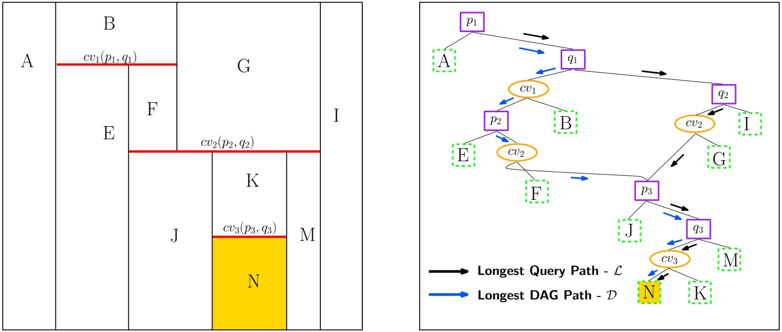
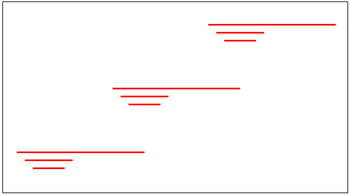


Towards a Worst-Case $\mathcal{D}/\mathcal{L}$ Ratio

- Top-to-bottom insertion order
- $\sqrt{n}$ blocks
- $\sqrt{n}$ segments in each block
- $\mathcal{D}$ is $\Omega(n)$
- $\mathcal{L}$ is $O(\sqrt{n})$



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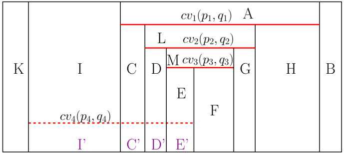

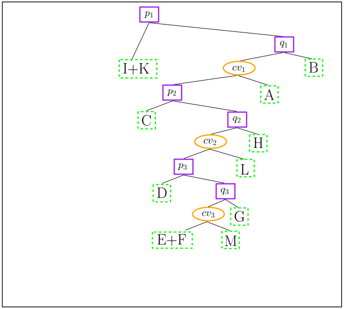

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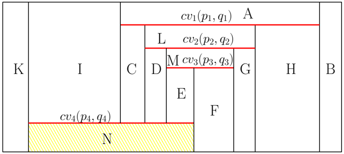

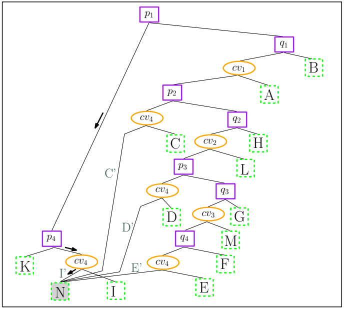

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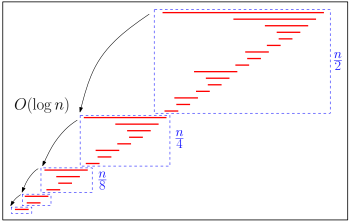
Towards a Worst-Case $\mathcal{D}/\mathcal{L}$ Ratio

- This construction ensures that each newly inserted segment intersects the trapezoid with the largest depth
- A query can skip an entire block using only one comparison
- Within the relevant block there are at most $O(\sqrt{n})$ comparisons

Worst-Case $\mathcal{D}/\mathcal{L}$ Ratio

- Top-to-bottom insertion order
- $\mathcal{D}$ is $\Omega(n)$
- $\mathcal{L}$ is $O(\log n)$
  - Achieved due to the recursive structure



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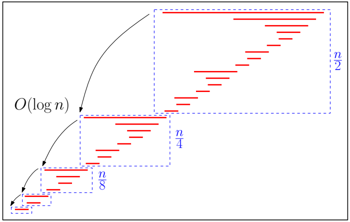

Theorem 1

The worst-case ratio between $\mathcal{D}$ and $\mathcal{L}$ is $\Omega(n/\log n)$ and this bound is tight.

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Verifying $\mathcal{L}$ After Construction in $O(n \log n)$ time

By verifying $\mathcal{L}$ in $O(n \log n)$ time we get the following expected $O(n \log n)$ time construction algorithm:

- Construct the DAG with some random insertion order
  - Verify $\mathcal{S}$ on the fly (can be accessed in $O(1)$ time)
  - Abort and rebuild if $\mathcal{S} \geq c_1 n$
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- A reduction from the collection of all trapezoids to a collection of axis-aligned rectangles
  - Uses a total order according to which curves can be translated one by one to $y = -\infty$ without hitting other curves that have not been moved yet [Guibas & Yao '80]
  - Can be computed in $O(n \log n)$ time [Ottmann & Widmayer '83]

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  - Can be computed in $O(n \log n)$ time [Ottmann & Widmayer '83]
- An $O(n \log n)$ time algorithm for computing the cover-depth of a collection of n axis-aligned rectangles [Alt & Scharf '10]

An $O(n \log n)$ Verification Algorithm for $\mathcal{L}$

The key ingredient:

Observation (Har-Peled)

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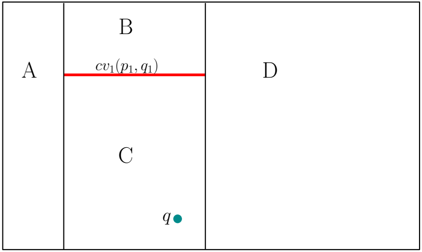

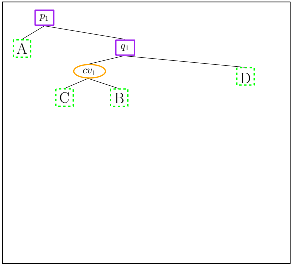

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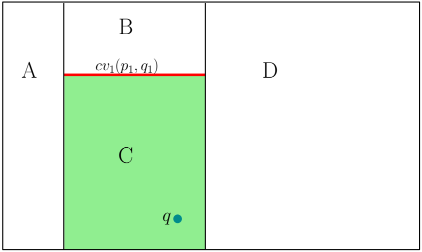

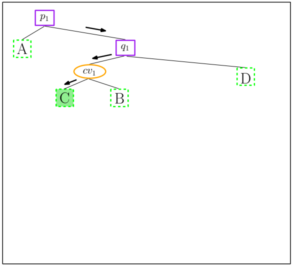

An $O(n \log n)$ Verification Algorithm for $\mathcal{L}$

The key ingredient:

Observation (Har-Peled)

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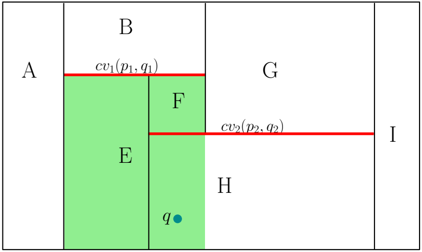

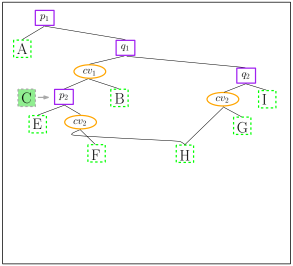

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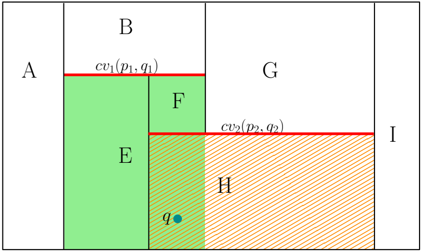

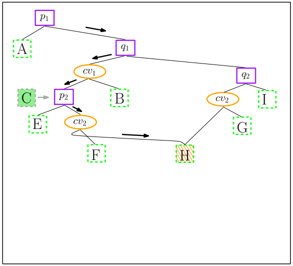

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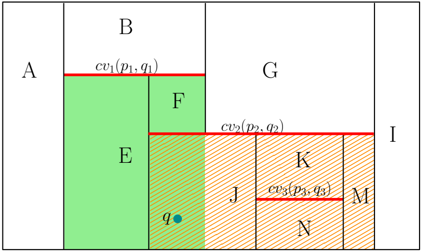

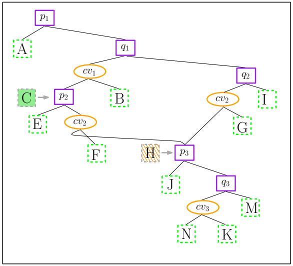

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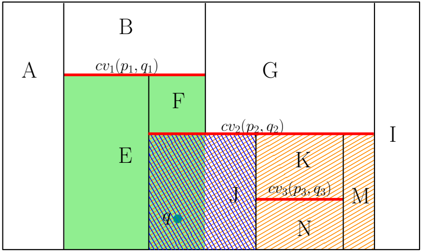
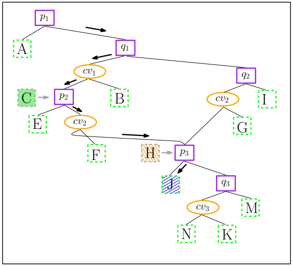
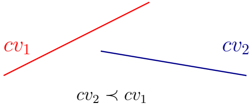
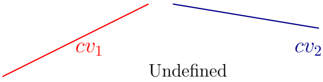




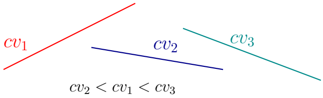
Reducing $\mathcal{T}^*$ to $\mathcal{R}^*$

- C - a set of interior disjoint x-monotone curves
- Define a total order $<$ as follows [Guibas & Yao '80]:
  - $\prec$ - an acyclic relation on C
    $cv_i \prec cv_j \Leftrightarrow cv_i(x) < cv_j(x)$ for some $x \in x\text{-range}(cv_i) \bigcap x\text{-range}(cv_j)$





  - Extend $\prec^+$ (the transitive closure of $\prec$) to a total order $<$:
    $cv_i < cv_j \Leftrightarrow (cv_i \prec^+ cv_j) or (\neg(cv_j \prec^+ cv_i) and (cv_i \text{ left } cv_j))$



Reducing $\mathcal{T}^*$ to $\mathcal{R}^*$

- $Rank : C \rightarrow \{1, ..., n\}$ - returns the order of $cv \in C$ when sorting C according to $<$

A trapezoid $t \in \mathcal{T}^*$ is reduced to a rectangle $r \in \mathcal{R}^*$, s.t.:

- t and r have the same x-range
- top and bottom edges of r lie on $y = Rank(top(t))$ and $y = Rank(bottom(t))$, respectively

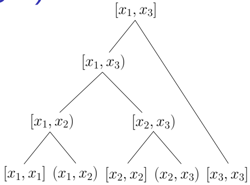
Showing that the Reduction Preserves the Depth

- Partition the plane into regions $Regions(arr)$ by passing a vertical line through every endpoint of the arrangement arr.
- For any region $a_t \in Regions(\mathcal{A}(\mathcal{T}^*))$ the matching rectangular region is $a_r \in Regions(\mathcal{A}(\mathcal{R}^*))$

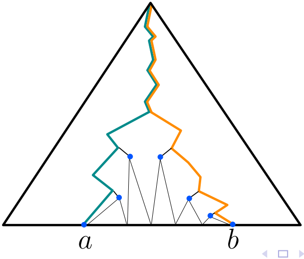
It can be shown that:

1. $Regions(\mathcal{A}(\mathcal{R}^*))$ spans the plane
2. For every $t \in \mathcal{T}^*$ covering a_t its reduced rectangle $r \in \mathcal{R}^*$ covers a_r
3. For every $r \in \mathcal{R}^*$ covering a_r its original trapezoid $t \in \mathcal{T}^*$ covers a_t

Computing the Depth of $\mathcal{A}(\mathcal{R}^*)$ in $O(n \log n)$ Time



- Algorithm by Alt & Scharf (2010)
- Basic data structure:
  a balanced binary tree for the intervals
- Keep *coverage* and *max-coverage* in every node
- Sweep from $y = +\infty$ to $y = -\infty$
- Sweep-line event: rectangle starts or ends
- Update a rectangle event with x-interval (a, b) in $\sim 2 \log n$ time



An $O(n \log n)$ Verification Algorithm for $\mathcal{L}$

Lemma 4

The length $\mathcal{L}$ in a linear size DAG can be verified in $O(n \log n)$ time.

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The length $\mathcal{L}$ in a linear size DAG can be verified in $O(n \log n)$ time.

Theorem 2

A point location data structure for a planar subdivision with n edges, which has $O(n)$ size and $O(\log n)$ query time in the worst case, can be built in expected $O(n \log n)$ time.

A Simpler Verification Algorithm for $\mathcal{L}$

We also suggest a randomized verification algorithm which:

- Runs in expected $O(n \log n)$ time
- Is much simpler
- Uses the existing structures (the DAG)

Outline

- Trapezoidal-map RIC point-location variants
- Depth vs. maximum query path length
- An efficient construction algorithm for static settings
- Open Problems

A Major Open Problem

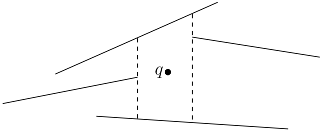
- Suppose that the structure is rebuilt whenever either the depth $\mathcal{D}$ or the size $\mathcal{S}$ exceed some thresholds.
- Can we still expect a constant number of rebuilds?

The End

Appendix

The Expected Query Time

- **Observation:** the depth of the DAG increases by at most 3 in every iteration
- Consider the path for a query q
  Lemma 5: Given a DAG for i segments, the probability that the removal of a segment will destroy the trapezoid containing q is at most $4/i$



(proof by figure)

The Expected Query Time

Bounding the expected length of the query path to q using backwards analysis:

- X_i - the number of nodes added to the path to q in iteration i
- P_i - the probability that there's a node on the path to q that is created in iteration i

$$\mathbb{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbb{E}[X_i] \leq \sum_{i=1}^{n} 3\mathrm{P}_i \leq \sum_{i=1}^{n} \frac{12}{i} = 12 \sum_{i=1}^{n} \frac{1}{i} = O(\log n)$$

The Expected Size

expected # DAG nodes = # leaves + expected # inner nodes

- Bounding # leaves:
  - # leaves = # trapezoids in the trapezoidal map
  - # trapezoids in the trapezoidal map is at most $3n + 1 = O(n)$
    - ★ The left side of every trapezoid is defined by a vertex (the left trapezoid is bounded by the boundary)
    - ★ Each curve defines the left of 2 trapezoids with its left endpoint and 1 trapezoid with its right endpoint

The Expected Size

- Bounding the expected # inner nodes created in the ith iteration:
  - Is at most the expected # new trapezoids created in the ith iteration (N_i)
  - $\mathcal{T}(S_i)$ - trapezoidal map for the first i inserted curves

$$N_i = \frac{1}{i} \sum_{c \in \mathcal{T}(S_i)} (\#\text{ trapezoids in } \mathcal{T}(S_i) \text{ that disappear by removing } c)$$

$$\leq \frac{1}{i} 4(\#\text{ trapezoids in } \mathcal{T}(S_i)) = \frac{O(i)}{i} = O(1)$$

- Summing up over all i $\rightarrow$ we get: expected $O(n)$ inner nodes in the final DAG

Corollary: expected # nodes is $O(n)$

The Expected Preprocessing Time

- Expected time to insert the ith curve: $O(\log i)$
  - Expected time to locate the left endpoint of the ith curve: $O(\log i)$
  - Expected # trapezoids created by the ith insertion: $O(1)$
- Each insertion takes at most expected $O(\log n)$ time
  $\Rightarrow O(n \log n)$ expected preprocessing time

A Tail Estimate

We show: the probability that the maximum query time is bad is very small

Lemma 1: Given a set S of n non-crossing line segments, a query point q, and a parameter $\lambda > 0$, the probability that the search path for q in the DAG has more than $3\lambda \log(n+1)$ nodes is at most $1/(n+1)^{\lambda \log 1.25 - 1}$

Proof sketch:

- Define a DAG with one source and one sink, the paths correspond to the permutations of S
  - A node for every subset of S, grouped in layers according to cardinality
  - A node in layer i has i incoming edges from nodes in layer $i-1$ and $n-i$ outgoing edges to nodes in layer $i+1$
  - An edge is marked if its insertion at that point changes the trapezoid containing q
  - Backwards analysis argument: at most 4 segments change the trapezoid containing q when they are removed from the subset
  - Therefore, any node has at most 4 marked incoming arcs (we always mark exactly 4)

A Tail Estimate

Proof sketch - continued:

- Finding the expected # marked edges on a path in the DAG
- X_i - (random variable) $= 1$ if the i-th arc on the path in the DAG is marked
- # nodes in the path is at most $3Y$, where $Y := \sum_{i=1}^{n} X_i$
- Using Markov's inequality:

$$\Pr[Y \geq \lambda \log(n+1)] = \Pr[e^{tY} \geq e^{t\lambda \log(n+1)}] \leq e^{-t\lambda \log(n+1)} \mathbb{E}[e^{tY}]$$

- Since the random variables are independent:

$$\mathbb{E}[e^{tY}] = \mathbb{E}[e^{\sum_{i=1}^{n} tX_i}] = \mathbb{E}[\prod_{i=1}^{n} e^{tX_i}] = \prod_{i=1}^{n} \mathbb{E}[e^{tX_i}]$$

- $\prod_{i=1}^{n} \mathbb{E}[e^{tX_i}] \leq \frac{2}{1}\frac{3}{2} \cdots \frac{n+1}{n} = n+1$, for $t = \log 1.25$

$$\Pr[Y \geq \lambda \log(n+1)] \leq 1/(n+1)^{\lambda t - 1}$$

A Tail Estimate

Lemma 2: Given a set S of n non-crossing line segments, and a parameter $\lambda > 0$, the probability that the maximum length of a search path in the DAG is more than $3\lambda \log(n+1)$ is at most $2/(n+1)^{\lambda \log 1.25 - 3}$

Proof sketch:

- Extend vertical walls at each endpoint- defining at most $2(n+1)^2$ regions
- Consider the search paths of representative points of these regions
- By Lemma 1 we get the required result

Size

We show: the probability that the size is bad is very small

Lemma 3: Given a set S of n non-crossing x-monotone curves, and a parameter $\rho \geq 1$, the probability that the size $\mathcal{S}$ of the DAG is greater than $15\rho n$ is at most $1/\rho$

Proof sketch:

- $\mathcal{C}$ - random variable representing the number of DAG nodes
- $\mathcal{C} = \#$ leaves $+$ sum of inner nodes created in iterations $1, ..., n$
- $\#$ leaves $= |\mathcal{T}(S)| \leq 3n + 1$
- $\#$ inner nodes created in iteration $i = \#$ new trapezoids created in iteration i minus $1 = k_i - 1$

Size

$$\mathbb{E}[\mathcal{C}] = \mathbb{E}[|\mathcal{T}(S)| + \sum_{i=1}^{n}(k_i - 1)] = \mathbb{E}[|\mathcal{T}(S)|] + \mathbb{E}[\sum_{i=1}^{n}(k_i - 1)]$$
$$\leq (3n+1) + \mathbb{E}[\sum_{i=1}^{n} k_i] - n = 2n + 1 + \mathbb{E}[\sum_{i=1}^{n} k_i] = 2n + 1 + \sum_{i=1}^{n} \mathbb{E}[k_i]$$

$$\mathbb{E}[k_i] \leq \frac{4 \cdot |\mathcal{T}(S_i)|}{i} \leq \frac{4(3i+1)}{i} = 12 + \frac{4}{i}$$

Therefore, $\mathbb{E}[\mathcal{C}] \leq 14n + 1 + 4H_n$
and $\mathbb{E}[\mathcal{C}] < 15n$, for $n \geq 12$

Using Markov's inequality: $\Pr[\mathcal{C} \geq 15\rho n] \leq \frac{\mathbb{E}[\mathcal{C}]}{15\rho n} = \frac{15n}{15\rho n} = \frac{1}{\rho}$