Triangulation

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Mar. 3rd, 2014

Outline

1 Triangulation
Polygon Terms and Definitions
The Art Gallery
Regularization and Triangulation
Literature

Application: Art Gallery

Given the floor plan of an art gallery modeled as a simple polygon with n vertices. Find out how many (and where) guards are needed to see the entire gallery, where each guard is stationed at a fixed point, has 360° vision, and cannot see through the walls.

Problem posed to Vasek Chvatal by Victor Klee at a math conference in 1973. Chvatal solved it quickly with a complicated proof, which has since been simplified significantly using triangulation.

Polygon Terms and Definitions

Definition (Polygon)
A polygon is a region of the plane bounded by a finite collection of line segments forming a simple close curve.

Theorem (Jordan Curve Theorem)
If C is a simple closed curve in \( \mathbb{R}^2 \), then \( \mathbb{R}^2 \setminus C \) has two components (an "inside" and "outside"), with C the boundary of each.

Definition (Simple Polygon)
A polygon is said to be simple (or Jordan) if it is enclosed by a single closed polygonal chain that does not cross itself. In particular, the polygon edges are pairwise disjoint in their interior and the degree of all vertices is two.

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Triangulation

Polygon Terms & Definitions (Cont.)
The chain \( v_1, v_2, \ldots, v_n \) defines a simple polygon iff
1. The segments \( s_1 = v_1v_2, s_2 = v_2v_3, \ldots, s_{n-1} = v_{n-1}v_n, s_n = v_nv_1 \) are disjoint in their interior.
2. Consecutive segments intersect only in their endpoints. Namely \( s_i \cap s_{i+1} = v_{i+1}, i = 1, 2, \ldots, n-1 \) and \( s_n \cap s_1 = v_1 \)
3. Non adjacent segments do not intersect \( s_i \cap s_j = \emptyset, j > i + 1 \).

P — a simple polygon.
\( \partial P \) — the boundary of P.
\( \partial P \subseteq P \), P is closed and contains its boundary.
By convention the vertices of a polygon are ordered counterclockwise around the interior of the polygon.
Interior of polygon is to the left of the boundary.

Application: Art Gallery

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Art Gallery: Lower Bound

Definition (Seeing)
A (guard) point \( p \) sees points \( q \in P \) if \( pq \subseteq P \).

Definition (Covering)
A set of guards \( G \) covers a polygon \( P \) if for any point \( p \in P \) there is a guard \( g \in G \) that sees \( p \).

- \( g(P) \) — minimum number of guards guarding \( P \).
- Cardinality of smallest set that covers \( P \).
- \( P_n \) — set of all simple polygons with \( n \) vertices.
- \( G(n) = \max_{P \in P_n} g(P) \) — maximum number of guards needed to guard a simple polygon with \( n \) vertices.
- \( G(n) \geq \lceil n/3 \rceil \)

Art Gallery: Upper Bound

- A diagonal of a polygon \( P \) is a segment connecting two vertices of \( P \) that strictly see each other.
- A triangulation is a partition of \( P \) into triangles formed by repeatedly inserting diagonals into \( P \).
- A vertex is strictly convex if its interior angle \( \alpha < \pi \).
- The interior angle of a reflex convex is \( \alpha > \pi \).
- Every polygon has at least one strictly convex vertex.
- Every polygon with \( n > 3 \) vertices has a diagonal.
- Every polygon may be partitioned into triangles by the addition of \( (0 \text{ or more}) \) diagonals.
- Proof by induction.
- \( T \) — a triangulation of a polygon \( P \) of \( n \) vertices.
- \( T \) uses \( n - 3 \) diagonals and consists of \( n - 2 \) triangles.

Art Gallery: Minimum Number of Guards

- A 3-coloring of the vertices yields 3 guards.
- However, the polygon can be guarded by only 2 guards.
- Finding the minimum number of guards is NP-hard.

Problem (Art Gallery Decision)
Given both a polygon and a number \( k \), determine whether the polygon can be guarded with \( k \) or fewer guards.

- Even the decision problem and all of its standard variations (such as restricting the guard locations to vertices or edges of the polygon) is NP-hard.

Art Gallery in \( \mathbb{R}^3 \)

- Even \( n \)-vertex guards do not suffice!
- Different triangulations can have different number of tetrahedra.
- Determining whether a polyhedron requires Steiner vertices for triangulation is NP-Complete.
  - Smallest example of a polyhedron that cannot be triangulated without adding new vertices. (Schoenhardt [1928]).
  - Every 3D polyhedron with \( n \) vertices can be triangulated with \( O(n^2) \) tetrahedra. [Cha84]

Art Gallery in \( \mathbb{R}^3 \) (Cont.)

- \( G(T) \) is a tree with a vertex degree at most 3.
- \( 3 \) consecutive vertices \( u, v, w \), form an ear if \( uvw \) is a diagonal
  - \( v \) is the ear tip.
- Every polygon of \( n > 3 \) has at least 2 non-overlapping ears.
- The graph of the triangulation \( T(P) \) is three-colorable.
- Every simple polygon \( P \) with \( n \) vertices can be guarded using \( \lfloor n/3 \rfloor \) guards; \( G(n) \leq \lfloor n/3 \rfloor \).
- Compute the triangulation of \( P \).
- Compute a 3 coloring for \( T(P) \).
- Choose the smallest set of vertices with the same color.
  - Its cardinality must be \( \lfloor n/3 \rfloor \).

Outlier

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  - Polygon Terms and Definitions
  - The Art Gallery
  - Regularization and Triangulation
  - Literature
Triangulation History

- Check all $n^2$ choices for a diagonal, each in $O(n)$ time. Repeat this $n - 1$ times, $O(n^2)$.
- Find an ear in $O(n)$ time; then recurse, $O(n^4)$ time.
- First non-trivial algorithm: $O(n \log n)$.
- A long series of papers and algorithms in 80s until Chazelle produced an optimal $O(n)$ algorithm. [Cha91]
- Linear time algorithm insanely complicated; there are randomized, expected linear time algorithm that are more accessible.

Regularization and Triangulation Algorithm Outline

**Definition (Monotone Polygonal Chain)**
A polygonal chain $C$ is monotone w.r.t. line $L$ if any line orthogonal to $L$ intersects $C$ in at most one point. [GJP+78]

**Definition (Monotone Polygon)**
A polygon is monotone w.r.t. $L$ if it can be decomposed into two chains, each monotone w.r.t. $L$.

- Partition polygon into trapezoids.
- Use trapezoids to make a monotone subdivision.
- Triangulate each monotone piece.

Partitioning a Polygon into Monotone Pieces

- At each vertex, extend vertical line until it hits a polygon edge.
- Each face of this decomposition is a pseudo trapezoid.
- Use plane sweep algorithm.
  - Time complexity is $O(n \log n)$.
- For each split (resp. merge) vertex $v$, add a diagonal that connects $v$ to the vertex of its left (resp. right) trapezoid.

**Vertex Ontology**
- Merge vertex
- Split vertex
- Start vertex
- End vertex
- Regular vertex

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Vertex Ontology

1. $\bullet$ — merge vertex
2. $\circ$ — split vertex
3. $\triangledown$ — start vertex
4. $\square$ — end vertex
5. $\Box$ — regular vertex

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Triangulation 20

Triangulation 21

Triangulation 22

Triangulation 23

Triangulation 24
Triangulation 25

Regularization and Triangulation Algorithm Proof

A polygon is x-monotone if it has no split vertices and no merge vertices.

Lemma (x-monotone)

- Suppose $P$ is a non x-monotone polygon.
- We have to prove that $P$ contains a split or a merge vertex.
- There are two cases
  - $p = r \Rightarrow \exists$ merge vertex
  - $p \neq r \Rightarrow \exists$ split vertex

Triangulation 26

Triangulation 27

Triangulation 28

Triangulation 29

Triangulation 30
Triangulating Monotone Polygons

1. Sort the vertices in lexicographically increasing order to yield $v_1, v_2, \ldots, v_n$.
2. Initialize a stack $\Gamma$, push($v_1, v_2$).
3. for $i = 3, \ldots, n$ do
4. if $v_i$ and $v_{i-1}$ are on opposite sides of $v_{i-2}$
5. Add diagonal $v_{i-2}v_i$.
6. if $v_{i-1}$ and $v_i$ are on opposite sides of $v_{i-2}$
7. Add diagonal $v_{i-2}v_{i-1}$.
8. else
9. Add diagonal from $v_i$ to all vertices on stack.
10. Add diagonal from $v_i$ to all vertices on stack.
11. end
12. end

Case I Case II

$\Gamma : v_{bot}, \ldots, v_{top} \Gamma : v_{bot}, \ldots, v_{top}$

11. Clear the stack.
12. Triangulate $P$.

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Regularization and Triangulation Algorithm Complexity

- Regularization via plane sweep takes $O(n \log n)$ time.
- Triangulation
  - Sorting by merging the two monotone chains of $P$ takes $O(n)$ time.
  - A vertex is added to stack once. Once it's visited during a scan, it's removed from the stack.
  - In each step, at least one diagonal is added; or the reflex stack chain is extended by one vertex.
- Triangulating a monotone polygon takes $O(n)$ time.
- Total time for polygon triangulation is therefore $O(n \log n)$.
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