Exercise 4.1  Give an example of a set of \( n \) points in the plane, and a query rectangle for which the number of nodes of the kd-tree visited is \( \Omega(\sqrt{n}) \).

Exercise 4.2  The algorithm we saw in class for searching in a kd-tree (where the search is guided by comparing the region of a node with the query region) can also be used when querying with ranges other than rectangles. For example, a query is answered correctly if the range is a triangle.

(a) Show that the query time for range queries with triangles is linear in the worst case, even if no points are reported at all. Hint: Choose all the input points to lie on the line \( y = x \).

(b) Suppose that a data structure is needed that can answer triangular range queries but only for triangles whose edges are horizontal, vertical or have slope \( +1 \) or \(-1\). Devise a linear-size data structure that answers such queries in \( O(n^{3/4} + k) \) time, where \( k \) is the number of points to be reported. Hint: Choose 4 coordinate axes in the plane and use a “4-dimensional” kd-tree.

(c) Improve the query time to \( O(n^{2/3} + k) \).

Exercise 4.3  Consider the orthogonal range-search structures that we studied in class for a set of points \( P \) in \( \mathbb{R}^d \), for a fixed \( d \). We wish to efficiently use them to answer membership queries, namely, to determine whether a point \( q \in \mathbb{R}^d \) is in \( P \).

(a) What is the time bound for such a query in a kd-tree?

(b) Suppose that a data structure is needed that can answer range queries but only for ranges whose edges are horizontal, vertical or have slope \( +1 \) or \(-1\). Devise a linear-size data structure that answers such queries in \( O(n^{3/4} + k) \) time, where \( k \) is the number of points to be reported. Hint: Choose 4 coordinate axes in the plane and use a “4-dimensional” kd-tree.

(c) Improve the query time to \( O(n^{2/3} + k) \).

Exercise 4.4  Given a \( y \)-monotone polygon \( P \) as an array of its \( n \) vertices in sorted order along the boundary. Show that, given a query point \( q \), it can be tested in time \( O(\log n) \) whether \( q \) lies inside \( P \).

Exercise 4.5  Design an algorithm with running time \( O(n \log n) \) for the following problem: Given a set \( P \) of \( n \) points, determine a value of \( \varepsilon > 0 \) such that the shear transformation \( \Phi : (x, y) \to (x + \varepsilon y, y) \) does not change the order (in \( x \)-direction) of points with unequal \( x \)-coordinates.

Exercise 4.x, bonus  Give a randomized algorithm to compute all pairs of intersecting segments in a set of \( n \) line segments in the plane in expected time \( O(n \log n + k) \), where \( k \) is the total number of intersections among the segments.