# The Arcs on the Beach Line

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# Overview

- The complexity of the lower envelope
- Arc appearance
- Arc disappearance

Figures taken from Chapter 9 in Computational Geometry Algorithms and Applications, Delaunay Trinagulations, by de Berg et al

# Complexity of the beach line

# How many arcs can appear on the beach line

- this in turn will determine the storage requirement of the status structure
- [arc: a maximal contiguous portion of a parabola on the lower envelope]
- $n$ sites: at most $2n - 1$ arcs
- more generally, the lower envelope of $n$ parabolas can have at most $2n - 1$ maximal arcs
- Demo: http://www.raymondhill.net/voronoi/rhill-voronoi.html

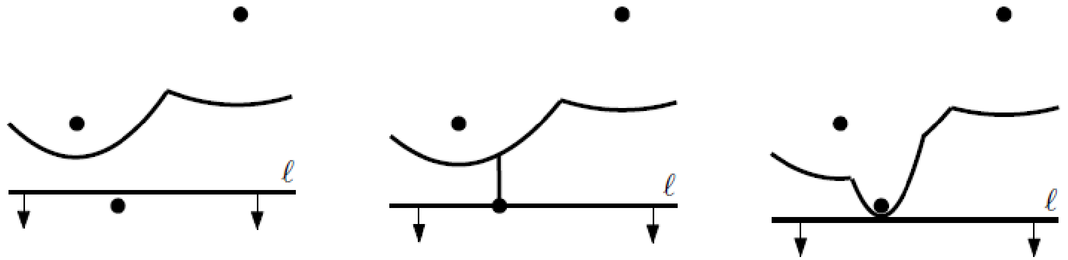
# Arc appearance and disappearance

# Reminder

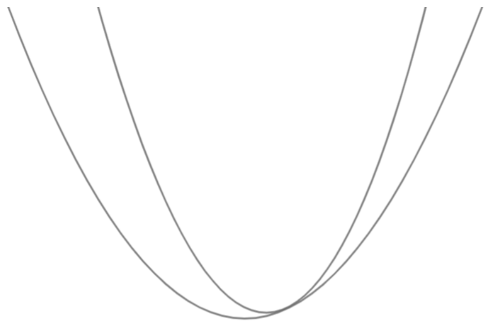
- the only way for a new parabolic arc to appear on the beach line is through a site event
- the only way for a parabolic arc to disappear from the beach line is through a circle event

# Arc appearance

- obviously possible at a site event




- a parabola cannot penetrate the beach line from above, I: if only one common point, the parabolas must be the same (simple algebra)



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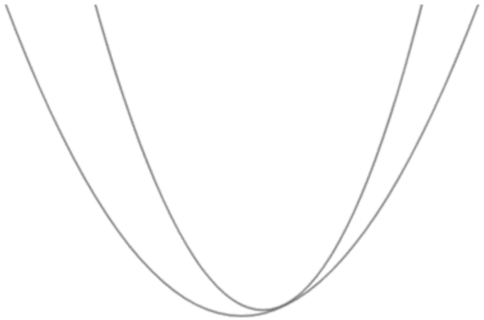




## The Appearance of a New Parabola on the Beach-Line

In Fortune's algorithm a parabola newly appears on the beach-line only in a site event, and only disappears from the beach-line in a circle event. Here we show that a parabola cannot penetrate the beach-line from above.

**Claim 1.** *In Fortune's sweep-line algorithm for constructing Voronoi diagrams, a parabola cannot penetrate the beach-line from above.*

*Proof.*
Let $s_1 = (x_1, y_1)$ and $s_2 = (x_2, y_2)$ be two distinct sites in the plane. Let $L(y)$ be the horizontal sweep line at height $y$.
Assume without loss of generality that the sweep-line reached $y = 0$. For a parabola to be active, its inducing site needs to be above the sweep-line. The parabola $\pi_i$ bounds from below all the points that are closer to $s_i$ than to any site $s_j$ lying below the current placement of the sweep-line $L(0)$. In particular it is the loci of points with equal distance to $s_i$ and $L(0)$:

$$(x - x_i)^2 + (y - y_i)^2 = y^2.$$

As $y_i$ is assumed to lie above $L(0)$ we can rewrite the equation of the parabola $\pi_i$ as follows:

$$\begin{aligned} x^2 - 2x_i x + x_i^2 + y^2 - 2y_i y + y_i^2 &= y^2, \\ 2y_i y &= x^2 - 2x_i x + x_i^2 + y_i^2, \\ y &= \frac{1}{2y_i}(x^2 - 2x_i x + x_i^2 + y_i^2). \end{aligned}$$

In summary the parabola $\pi_i$ defined by $s_i$ when the sweep line is at $y = 0$ is $\frac{1}{2y_i}(x^2 - 2x_i x + x_i^2 + y_i^2)$.
Let $\pi_{12} = \pi_1 - \pi 2$. That is, $\pi_{12}$ equals

$$x^2(\frac{1}{2y_1} - \frac{1}{2y_2}) + x(\frac{x_2}{y_2} - \frac{x_1}{y_1}) + (\frac{x_1^2 + y_1^2}{2y_1} - \frac{x_2^2 + y_2^2}{2y_2}).$$

The roots of the parabola $\pi_{12}$ are the $x$ coordinates of the intersection points between $\pi_1$ and $\pi_2$. Consider the equation $\pi_{12} = 0$, rewritten as $Ax^2 + Bx + C$ with

$$\begin{aligned} A &= \frac{1}{2y_1} - \frac{1}{2y_2}, \\ B &= \frac{x_2}{y_2} - \frac{x_1}{y_1}, \\ C &= \frac{x_1^2 + y_1^2}{2y_1} - \frac{x_2^2 + y_2^2}{2y_2}. \end{aligned}$$

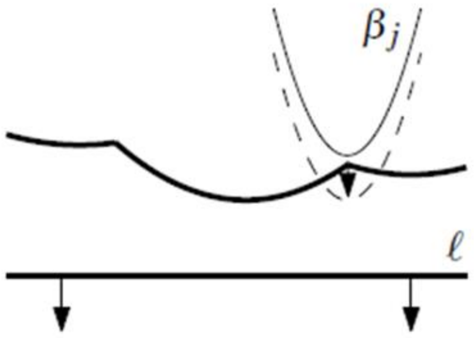
The discriminant of the equation $\pi_{12} = 0$ is

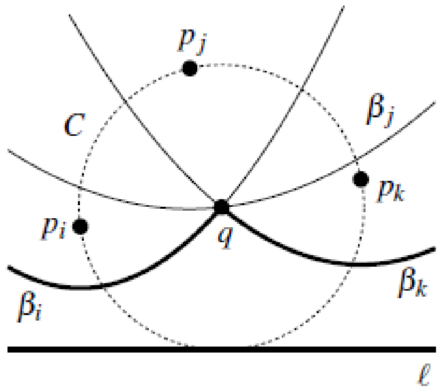
$$\begin{aligned} B^2 - 4AC &= (\frac{x_2}{y_2} - \frac{x_1}{y_1})^2 - 4(\frac{1}{2y_1} - \frac{1}{2y_2})(\frac{x_1^2 + y_1^2}{2y_1} - \frac{x_2^2 + y_2^2}{2y_2}) \\ &= (\frac{x_2}{y_2} - \frac{x_1}{y_1})^2 - (\frac{1}{y_1} - \frac{1}{y_2})(\frac{x_1^2 + y_1^2}{y_1} - \frac{x_2^2 + y_2^2}{y_2}) \\ &= \frac{x_2^2}{y_2^2} - 2\frac{x_1 x_2}{y_1 y_2} + \frac{x_1^2}{y_1^2} - (\frac{x_1^2 + y_1^2}{y_1^2} - \frac{x_2^2 + y_2^2}{y_1 y_2} - \frac{x_1^2 + y_1^2}{y_1 y_2} + \frac{x_2^2 + y_2^2}{y_2^2}) \\ &= \frac{x_2^2}{y_2^2} - 2\frac{x_1 x_2}{y_1 y_2} + \frac{x_1^2}{y_1^2} - \frac{x_1^2}{y_1^2} - 1 + \frac{x_1^2 + y_1^2 + x_2^2 + y_2^2}{y_1 y_2} - \frac{x_2^2}{y_2^2} - 1 \\ &= \frac{1}{y_1 y_2}(x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2x_1 x_2 - 2y_1 y_2) \\ &= \frac{1}{y_1 y_2}[(x_1^2 - 2x_1 x_2 + x_2^2) + (y_1^2 - 2y_1 y_2 + y_2^2)] \\ &= \frac{1}{y_1 y_2}[(x_1 - x_2)^2 + (y_1 - y_2)^2] \end{aligned}$$

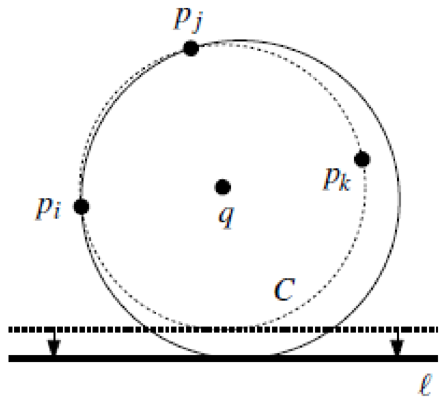
Notice that $y_1 y_2 > 0$ by assumption. Therefore the final expression above is non-negative. It equals zero only if $x_1 = x_2$ and $y_1 = y_2$, in which case the parabolas are identical. It follows that for two distinct sites, both of which lying above the sweep line, the corresponding parabolas, no matter where the sweep line is (we can always translate the plane such that the sweep line is at $y = 0$), have two intersection points.
□

# Arc appearance, cont'd

- a parabola cannot penetrate the beach line from above, II, through a breakpoint

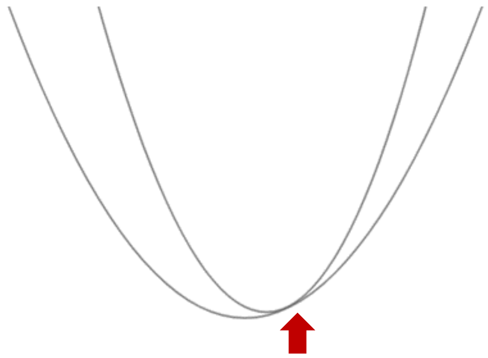


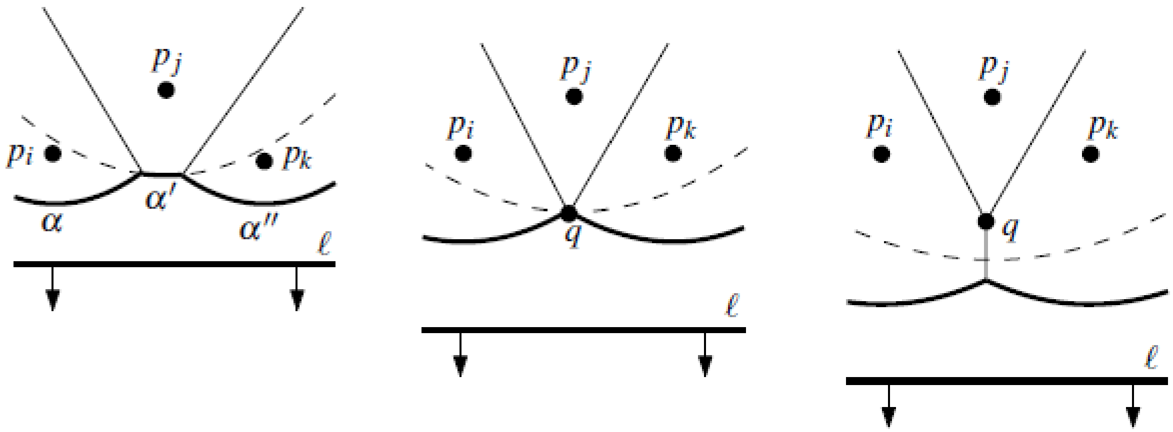



# Arc disappearance

- a parabola cannot disappear into another parabola–the same argument as before (considering the opposite direction)



- a parabola can only disappear between two other parabolic arcs



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