On Disjoint Concave Chains in Arrangements of (Pseudo) Lines—Corrigendum

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Abstract

In the paper mentioned in the title [1] we have asserted that the maximum number of edges of m pairwise-disjoint x-monotone concave polygonal chains, contained in the union of n lines or pseudo lines, is $\Theta(m^{2/3}n^{2/3}+n)$. While the assertion is correct, the analysis of [1] was incomplete, and hence erroneous. In this note we complete the analysis and thus obtain a correct proof of the assertion.

In this note we follow the notation set in the original paper [1]. Let $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ be a collection of n *pseudo lines* in the plane, which we define to be x–monotone unbounded curves, each pair of which intersect at most once. We order $\mathcal{L}$ by the reverse vertical order of the pseudo lines at $x = -\infty$ (so $\ell < \ell'$ if ℓ lies higher than ℓ' at $x = -\infty$). This is a linear order, and we may assume that this order is $\ell_1 < \ell_2 < \cdots < \ell_n$. Let us denote by $\mathcal{A} = \mathcal{A}(\mathcal{L})$ the *arrangement* of the pseudo lines in $\mathcal{L}$. The arrangement $\mathcal{A}(\mathcal{L})$ is the partition of the plane into cells of dimensions 0,1 and 2 induced by the pseudo lines in $\mathcal{L}$.

A *concave chain* c in $\mathcal{A}$ is an x–monotone (connected) path that is contained in the union of the pseudo lines of $\mathcal{L}$, such that the sequence of pseudo lines traversed by c from left to right is a strictly decreasing sequence. Informally, as we traverse c from left to right, whenever we reach a vertex of $\mathcal{A}$, we can either continue along the pseudo line we are currently on, or make a right (i.e. downward) turn onto the other pseudo line, but we cannot make a left (upward) turn; in case the pseudo lines are real lines, c is indeed a concave polygonal chain. It is clear that the number of turns along a concave chain is at most $n - 1$.

In [1] we have studied the problem of bounding the maximum joint combinatorial complexity, $E(m, n)$, of m *disjoint* concave chains in an arrangement of n pseudo lines. We argued there that $E(m, n) = \Theta(m^{2/3}n^{2/3} + n)$. The scheme of the upper

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bound proof, which we also follow in this note, is first to obtain an intermediate bound $O(mn^{1/2}+n)$ on $E(m,n)$ (referred to as a 'Canham bound'), and then to divide the problem into a collection of subproblems of a smaller size, to apply the intermediate bound to each subproblem, and to collect the sub-results into the global bound stated above.

For simplicity of presentation, we first consider the case of lines, and then comment on the modifications needed to handle the case of pseudo lines.

In our analysis in this note, we need to use an alternative approach to our "divide and conquer" process, namely, instead of applying random sampling, we use a $\frac{1}{r}$-*cutting* [3]. Given a set S of n lines in the plane, a $\frac{1}{r}$-cutting for S is a collection Ξ of (possibly unbounded) triangles which cover the plane, have pairwise disjoint interiors, and are such that the interior of each triangle in Ξ is intersected by at most $\frac{n}{r}$ lines of S. The existence of such a cutting with only $O(r^2)$ triangles has been shown, e.g., by Matoušek [3]. In compatibility with the original note, we will refer to these triangles as "funnels". We also assume that each funnel has a unique bottom edge, a unique top edge, and a vertical edge; this is achieved by splitting, if necessary, each funnel into two funnels by a vertical segment through its 'middle' vertex.

The error in the original paper is in Step 5 of the analysis—"Complexity of the internal subchains ('zones')". An internal subchain is a connected portion of the intersection of a chain with a funnel, which does not contain any endpoint of the full chain. We argued in [1] that if we fix a pair of edges of the i-th funnel, then the joint complexity of all the internal subchains that stretch between these two edges is $O(n_i)$, where n_i is the number of lines crossing the i-th funnel. This is indeed the case for most pairs of edges, but the argument in [1] fails for internal subchains that lie within a funnel and start and end on the bottom edge of the funnel. We refer to such internal subchains as *bottom subchains*; see Figure 1. Note that for any other edge of a funnel it is impossible that an internal subchain start and end on it, due to the concavity of the chains.

We thus need to analyze the overall complexity of all bottom subchains, and do it as follows. We construct the following (plane embedding of a) planar graph G. Its nodes are the bottom edges of the $O(r^2)$ funnels. If c is a chain that appears as a bottom subchain in two funnels f_1, f_2, and the portion γ of c between f_1 and f_2 does not contain any bottom subchain, we connect the bottom edges of f_1 and f_2 in G, and draw the connecting edge in the plane along γ.

Since the given chains are pairwise disjoint, it is clear that the resulting graph G is a planar graph with $k=O(r^2)$ nodes. By Euler's formula, the number of its edges is $O(k+z)$, where z is the number of faces with only 2 edges. Note that the number of bottom subchains is m plus the number of edges of G (along each chain, the number of edges of G is one less than the number of bottom subchains).

Next, fix a pair e_1, e_2 of bottom funnel edges, and consider all the faces of G formed between two edges connecting e_1 with e_2, and having only those 2 edges. Partition this set of faces into *clusters*, where a cluster is a sequence of such faces

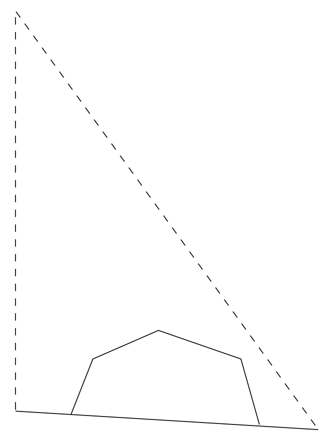

Figure 1: A bottom subchain

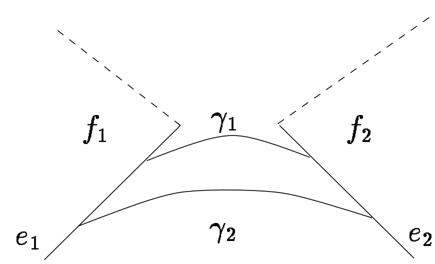

Figure 2: A face of G having only two edges; γ_1 lies above γ_2

that are adjacent to each other. We apply this clustering process to each pair of bottom funnel edges (e_1, e_2), and conclude, by an easy application of Euler's formula, that the overall number of clusters is $O(k) = O(r^2)$.

Let F be a face of G with only 2 edges that participates in one of these clusters. F is bounded by 4 arcs: (portions of) the bottom edges e_1, e_2 of two funnels f_1, f_2, and two portions γ_1, γ_2 of two respective chains c_1, c_2. Suppose γ_1 lies "above" γ_2 near e_1; formally, this is defined by requiring that the portion of e_1 between its intersections with γ_1 and γ_2, and the edge of γ_1 incident to e_1 have disjoint x-projections. It is easily checked that this condition must hold for exactly one of γ_1 or γ_2 because of the structure of the face F; see Figure 2 for an illustration).

By assumption, the continuation of the chain c_1 past γ_1 into f_1 forms there a bottom subchain c_1', whose other endpoint must therefore also lie on e_1. If this endpoint lies on the boundary of any other face F' of the same cluster, c_1 re-enters F' there and thus, as is easily seen, cannot leave F' again; see Figure 3 for an illustration. Since F' does not contain in its interior any edge of G, it follows that the remainder of c_1 within F' does not contain any bottom subchain, so the bottom subchain c_1' is an extreme such subchain along c_1. Thus the number of such subchains is at most

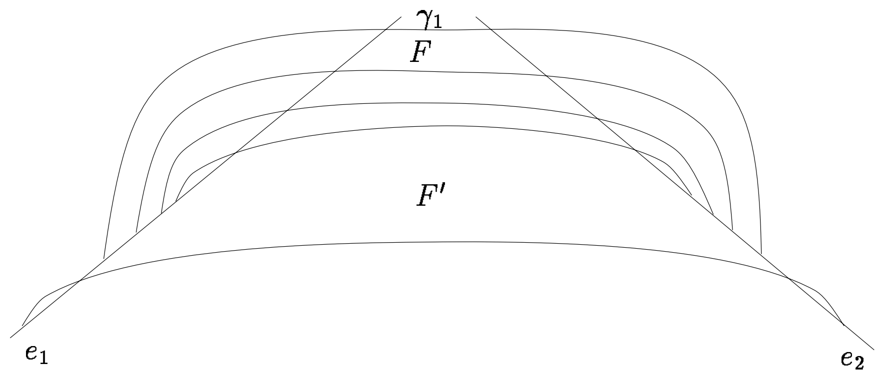

Figure 3: Extreme edges in a cluster

$2m$. If c_1' is indeed such an extreme subchain, we also call γ_1 an *extreme edge* of G (along c_1). It easily follows that if the above γ_1 is extreme, then the 'top' edges of all the faces of the cluster between F and F' are also extreme.

Let $(F_1, \ldots, F_t)$ be a cluster of 2-edge faces lying between two bottom funnel edges e_1 and e_2, and let γ_i be the edge of G separating between F_{i-1} and F_i, for $i = 2, \ldots, t$; we also define γ_1 to be the other edge of F_1, and γ_{t+1} to be the other edge of F_t. See Figure 4 for an illustration. We assume that γ_i is a portion of a chain c_i, for $i = 1, \ldots, t+1$, and that γ_i lies above γ_{i+1} near e_1 (in the sense defined above), for all $i \leq t$. Let f_1 be the funnel whose bottom edge is e_1. Now each non-extreme edge γ_i, when extended into f_1 as a bottom subchain, exits e_1 at a point that does not lie on the boundary of any face F_j (because otherwise, by the above argument, γ_i would have been an extreme edge). This is easily seen to imply that all the bottom subchains c_j', which are the extensions of the respective non-extreme edges γ_j into f_1, are *nested* within each other, that is, the portion of f_1 enclosed between c_i' and e_1 fully contains the similar portion defined for c_j', for all $1 \leq i < j \leq t+1$ for which γ_i and γ_j are non-extreme; see Figure 4.

To recap, we have shown that the set of bottom subchains, in all funnels, consists of at most $O(m)$ individual bottom subchains, plus $O(r^2)$ clusters, each consisting of some number of nested subchains within a single funnel.

We first estimate the overall complexity of the nested clusters. Let $(c_1', \ldots, c_q')$ be such a nested cluster of bottom subchains within a funnel f_i. It is easily verified that any line ℓ crossing f_i can contribute an edge to at most one subchain of the cluster. Hence the overall complexity of the subchains in the cluster is at most $n_i \leq n/r$, which implies that the overall complexity of all clusters is $O(nr)$.

The remaining $O(m)$ bottom subchains are distributed among the funnels, thereby creating $k = O(r^2)$ subproblems, where the i-th subproblem consists of m_i bottom subchains within the i-th funnel, in which at most n/r lines are involved. We use the 'Canham bound' mentioned above for each of these subproblems, as in [1], to obtain an overall complexity of $O(m(n/r)^{1/2}) + rn)$. Choosing $r = \Theta(m^{2/3}n^{-1/3})$, we get the

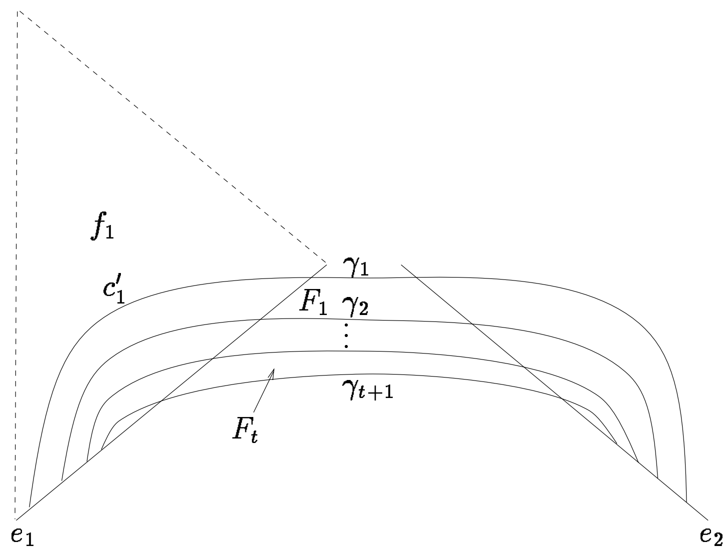

Figure 4: A cluster of faces and nested bottom subchains

asserted upper bound on $E(m, n)$. The lower bound is established in [1].

The case of pseudo lines is handled in much the same way. The construction of a $(1/r)$-cutting in an arrangements of lines, as given in [4], can be extended to arrangements of pseudo lines. This technique constructs various levels in the arrangement, and then 'shortcuts' each level by joining certain pairs of vertices on the same level by straight segments. These shortcuts can also be made in the case of pseudo lines, by applying Levi's lemma [2], which asserts that, given a collection $\mathcal{L}$ of pseudo lines and two points p, q which do not both lie on the same pseudo line, then there exists a curve γ passing through p and q so that the collection $\mathcal{L} \cup \{\gamma\}$ is also a collection of pseudo lines. (We also need to ensure that if all pseudo lines in $\mathcal{L}$ are x-monotone, γ can also be chosen to be x-monotone. While the original formulation of Levi's lemma does not assert this explicitly, one can show that this is indeed the case (Emo Welzl, private communication): Extend a vertical line through every intersection point of two pseudo lines in $\mathcal{L}$. Since each pseudo line in $\mathcal{L}$ is x-monotone, the pseudo lines of $\mathcal{L}$ together with the additional vertical lines still form a collection of pseudo lines. We apply Levi's lemma to this extended collection and deduce that the curve γ through the two points p and q is either x-monotone or can easily be made x-monotone, so that $\mathcal{L} \cup \gamma$ remains a collection of pseudo lines.) The rest of the construction is the same as in [4].

The resulting funnels in this construction are trapezoidal-like cells, each delimited by one or two vertical segments and by a top edge and a bottom edge, each being a portion of some pseudo line; see Figure 5. We can now apply the preceding analysis essentially verbatim to the resulting funnels. We leave it to the reader to verify that the analysis indeed extends to the case of pseudo lines.

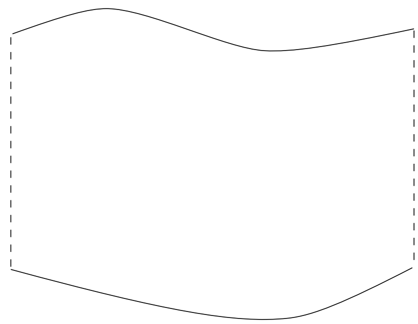

Figure 5: A funnel in the cutting for pseudo lines

In summary, we have:

Theorem 1 *The maximum joint combinatorial complexity, $E(m,n)$, of m disjoint concave chains in an arrangement of n lines or pseudo lines in the plane is $\Theta(m^{2/3}n^{2/3}+n)$.*

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