The Complexity of the Free Space for a Robot Moving Amidst Fat Obstacles *

A. Frank van der Stappen[†]
Dan Halperin[‡]
Mark H. Overmars[†]

Abstract

We propose a new definition of *fatness* of a geometric object and compare it with alternative definitions. We show that, under some realistic assumptions, the complexity of the free space for a robot with any fixed number of degrees of freedom moving in a d-dimensional Euclidean workspace with fat obstacles is linear in the number of obstacles. The complexity of motion planning algorithms depends, to a large extent, on the complexity of the robot's free space, and theoretically, the complexity of the free space can be very high. Thus, our result opens the way to devising efficient motion planning algorithms in certain realistic settings.

1 Introduction

It has been recently noted that, in certain problems in computational geometry, the relatively high complexity implied by worst-case lower bound constructions, can be avoided if we assume that the objects at hand have a certain "fatness" property. This paper discusses fatness in the context of algorithmic motion planning.

1.1 Background: motion planning and fatness

Autonomous robots are one of the ultimate goals in the field of robotics. An autonomous robot should accept high-level descriptions of tasks and execute these tasks

*Research is supported by the Dutch Organization for Scientific Research (N.W.O.) and partially supported by the ESPRIT II Basic Research Actions of the EC under contract No. 3075 (project ALCOM) and the ESPRIT III BRA Project 6546 (PROMotion).

†Department of Computer Science, Utrecht University, P.O. Box 80.089, 3508 TB Utrecht, The Netherlands.

‡Department of Computer Science, Tel Aviv University, 69978 Tel Aviv, Israel. (Current address: Robotics Laboratory, Department of Computer Science, Stanford University, Stanford, CA 94305, USA.)

with as little intervention from its environment as possible, and ideally without further intervention at all. A fundamental task for an autonomous robot would be to move from one place to another while avoiding collision with the obstacles on its way. The problem of finding such a collision-free motion is referred to as the *motion planning problem.* (A comprehensive overview of the state-of-the-art in robot motion planning is given in [8].) There are many variants to the motion planning problem and in this paper, we consider the following version:

> Given a robot $\mathcal{B}$ moving amidst a collection of obstacles $\mathcal{E}$, and an initial placement Z_0 and a desired final placement Z_1 for $\mathcal{B}$, find a continuous motion for $\mathcal{B}$ from Z_0 to Z_1 during which the robot avoids collision with the obstacles, or report that no such motion exists.

There is a large variety of approaches to solving motion planning problems [8]. One approach, followed mainly in theoretical computer science, aims to develop combinatorial, non-heuristic solutions to motion planning problems. This approach fits naturally in *computational geometry.* The computational geometry approach to robot motion planning is often referred to as *algorithmic motion planning.*

We denote the *workspace* where the robot moves by W. This workspace, or physical space, usually corresponds to the Euclidean space of dimension two or three, since these are the most interesting cases from a practical point of view. The collection of obstacles $\mathcal{E}$ is a closed subset of W.

The *configuration space*, i.e., the space of parametric representations of placements of the robot $\mathcal{B}$, is usually not the same as the workspace W. The dimension of the configuration space is determined by the number of degrees of freedom of $\mathcal{B}$. As an example, take a line segment ("ladder") moving in the plane. A placement of the ladder can be identified by the position of some reference point on the ladder in the plane and the orientation of the ladder. Hence, the configuration space is three-dimensional in this case. We distinguish three types of points in the configuration space of a specific problem according to the placements of the robot that they represent: *free placements,* where the robot does not intersect any obstacle, *forbidden placements,* where the robot intersects the interior of an obstacle, and *semi-free placements,* where the robot is in contact with the boundary of an obstacle, but does not intersect the interior of any obstacle.

The collection of all the points in the configuration space that represent semi-free robot placements partitions the configuration space into *free regions* and *forbidden regions.* In a motion planning problem with two degrees of freedom, for instance, the points representing semi-free placements of the robot lie on several curves in a two-dimensional configuration space. We will call these curves *constraint curves*. Note that each such curve is induced by the contact of a robot feature and an obstacle feature. We will use the term *feature* to describe a basic part on the boundary of a geometric object whether an obstacle or the robot. For example, the features of a polygonal robot will be the vertices and edges on its boundary.

We will refer to the set of free placements as the *free space*, and denote it by FP. We will call a maximal free region of the configuration space a *free cell.* The free space is the collection of free cells in the configuration space. The motion planning problem now reduces to the problem of finding a continuous path between an initial configuration and a goal configuration in FP (i.e., between two points in FP). We are therefore interested in studying a collection of cells—the free cells—in the partitioning of 2D space by a collection of constraint curves (for motion planning problems with two degrees of freedom), the collection of free cells in the partitioning of a three-dimensional configuration space by constraint surfaces (for motion planning problems with three degrees of freedom), and similarly, the collection of free cells in the partitioning of an f-dimensional configuration space by constraint hypersurfaces, for motion planning problems with f degrees of freedom. This is the point where robot motion planning overlaps a basic study in computational geometry, namely, the study of the combinatorial structure of *arrangements* of algebraic curves or surfaces in low-dimensional Euclidean spaces.

The *complexity of a cell* in an arrangement of hypersurfaces is defined to be the number of faces of various dimensions on its boundary. For example, the complexity of a face (a two-dimensional cell) in an arrangement of curves in the plane is the number of edges and vertices on its boundary, where a *vertex* is either an endpoint of a curve or the meeting point of two curves, and an *edge* is a maximal portion of a curve meeting no vertex of the arrangement. (For a detailed discussion on arrangements of curves and surfaces, and their connection to motion planning see, e.g., [4], [5].) The *complexity of the free space* is the sum of the complexities of the cells that comprise the free space, and, hence, bounded by the complexity of the entire arrangement of the constraint hypersurfaces. As each constraint hypersurface is induced by a contact of a robot feature and an obstacle feature, the intersection of j such surfaces corresponds to the simultaneous occurrence of j contacts for the robot. Thus, the complexity of the free space is determined by the number of different single and multiple contacts, since they determine the complexity of the arrangement of constraint hypersurfaces.

Different techniques exist to find a path between two points in FP. We refer the reader to the papers [16], [19] and the book [8] for extensive surveys of these techniques. We merely emphasize that the performance of many of the existing techniques depends, to a large extent, on the complexity of the free space of the corresponding problem. The complexity of the free space, in turn, is determined by the number of multiple contacts of the robot $\mathcal{B}$ and the obstacles. A multiple contact of the robot $\mathcal{B}$ is a placement in which it touches more than one obstacle feature. To get a feeling of what a multiple contact is, consider the case of a ladder *translating* among polygonal obstacles in the plane. This is a motion planning problem with two degrees of freedom and the constraint curves it induces in the 2D configuration space are straight line segments. Each of these constraint segments is induced either by the contact of a ladder endpoint with an obstacle edge, or by the contact of the interior of the ladder with an obstacle vertex. Consider now the case where each ladder endpoint touches a distinct obstacle edge (and assume further that these two edges are not parallel). The contact of each ladder endpoint with an obstacle edge is expressed as a segment in the configuration space, and this double contact will manifest itself as the meeting point of these two segments, namely as a vertex in the configuration space.

Unfortunately, in general, the number of multiple contacts, and hence, the complexity of the free space, can be very high. If n is the number of obstacle features and f is the number of degrees of freedom of the robot (i.e., the dimension of the configuration space) and the number of robot features is bounded by some constant, then this complexity can be $\Omega(n^f)$. So, theoretically, motion planning techniques whose performance depends on the size of the free space are very expensive. Fortunately, in many practical situations the complexity of the free space FP is much smaller and, hence, these methods might become feasible. A study of properties that limit the number of multiple contacts for the robot (and hence the complexity of FP) is therefore of obvious importance.

In many practical cases the relative positions and the shapes of the obstacles are such that the number of multiple contacts for the robot $\mathcal{B}$ is very low. Obstacles that are far apart clearly result in less double contacts for $\mathcal{B}$ than obstacles that are cluttered. Similarly, obstacles that have no long and skinny parts will induce less double contacts than obstacles that do have such parts.

As mentioned above, several authors have noted that, in certain problems in computational geometry, the relatively high complexity implied by worst-case lower bound constructions, can be avoided if we assume that the objects at hand have a certain "fatness" property; see, e.g., [1],[11],[14]. For example, if we consider the combinatorial complexity of the union boundary of a collection of triangles in the plane, whose angles are all greater than a fixed minimum angle δ, then it has been shown to be nearly-linear [11] (where the constant of proportionality in the big-Oh notation, depends on δ), whereas, in general, the union boundary of a collection of n triangles in the plane can have complexity $\Theta(n^2)$ in the worst case. Furthermore, "fatness" may lead to simple algorithms for solving problems on the fat objects; see, e.g., [1].

1.2 Summary of results

In this paper we propose a slightly different definition of fatness. We say that an object E is k-fat, if for any (hyper)sphere S with center inside E and whose boundary intersects E, the volume of E inside S is at least $\frac{1}{k}$th of the volume of S (see below, in Section 2 for more details). We compare this new definition of fatness with other possible definitions.

Using this definition of fatness we show that the complexity of the free space for a robot moving in a "fat" setting is linear. We assume that each obstacle is fat according to our definition of fatness, and make some additional realistic assumptions about the robot and the obstacle. The main result of the paper is summarized in the following theorem:

Theorem 4.1 *Let $\mathcal{E} \subseteq \mathbb{R}^d$ be a set of n k-fat obstacles of constant complexity. The diameter of the minimal enclosing hypersphere of each obstacle $E \subseteq \mathcal{E}$ is at least d_{min}. Let $\mathcal{B}$ be a robot of constant complexity with f degrees of freedom and with diameter $d_{\mathcal{B}} \leq b \cdot d_{min}$. For each j $(2 \leq j \leq f)$, the number of j-fold contacts of the robot $\mathcal{B}$ is linear in the number of obstacles: $\mathcal{O}(n)$.*

Note that the fatness of the obstacles alone is insufficient for obtaining the linear complexity result, and the theorem above also assumes a bound on the size of the robot relative to the obstacle with smallest enclosing hypersphere. (It also assumes a fixed descriptive complexity of the robot and each obstacle, but these are more standard assumptions in algorithmic motion planning.) The proof of the result is based on the following idea: we consider the smallest obstacle and prove a constant upper bound on the number of larger obstacles that lie close enough to this smallest obstacle so that both can be involved in a single multiple contact, and repeat this argument for every next smallest obstacle. Using this idea, we obtain a linear number of multiple contacts.

To see an immediate consequence of our result, consider the motion planning algorithm by Sifrony and Sharir [17] for a ladder translating and rotating among polygonal obstacles in the plane. The algorithm has running time $\mathcal{O}(K \log n)$, where K is the number of obstacle feature pairs that are less than the length of the ladder apart. Our result shows that the number K is linear in case of fat obstacles. As a consequence, the algorithm then runs in $\mathcal{O}(n \log n)$ time, whereas it might take $\mathcal{O}(n^2 \log n)$ time for non-fat obstacles. However, in general, our linear combinatorial complexity result does not immediately imply an efficient algorithm for the motion planning problem at hand. The discussion of the algorithmic aspect of motion planning among fat obstacles is postponed to a companion paper [18].

The remainder of the paper is organized as follows. In Section 2 we present our definition of fatness of a geometric object, and compare it with a few alternative definitions. In Section 3 we show that the number of larger fat objects in the proximity of a given objects is bounded by a constant. This result is used in Section 4 to show that, under some realistic assumptions, the free space of a robot moving in a workspace with fat obstacles has linear complexity. Some concluding remarks are given in Section 5.

2 Fat objects

The number of multiple contacts for a robot $\mathcal{B}$, and, hence, the complexity of the free space, is in most practical situations much lower than the worst case number of multiple contacts induced by the dimension of the configuration space. If, for example, the obstacles are far apart then the robot might not be able to touch two obstacles simultaneously, so there is no double contact at all. The complexity of the free space is only $\mathcal{O}(n)$ (the number of single contacts) in this case. Long and skinny obstacles can be involved in a larger number of multiple contacts than more "compact" obstacles without protuberances. These compact, or *fat*, obstacles will therefore result in a lower complexity of the free space.

Fatness turns out to be a very interesting property in computational geometry. Alt *et al.* [1] and Matoušek *et al.* [11] show that the upper bounds on the combinatorial complexity of the union of certain geometric figures are lower if these figures are fat. The union of geometric figures plays a role in many computational geometry applications. Fatness can lead to efficient algorithms. Overmars [14] shows that point location queries in fat subdivisions (no cell has long and skinny parts) in d-dimensional space can be performed in a simple way in $\mathcal{O}(\log^{d-1} n)$ time with a data structure that uses $\mathcal{O}(n \log^{d-1} n)$ storage.

Our definition of fatness in a d-dimensional Euclidean workspace involves d-dimensional closed hyperspherical regions centered at some arbitrary point in an object E. The closed hyperspherical region with radius r centered at m will be denoted by $S_{m,r}$, so

$$S_{m,r} = \{x \in \mathbb{R}^d | d(x,m) \leq r\};$$

the boundary of this region will be denoted by $\partial S_{m,r}$, so

$$\partial S_{m,r} = \{x \in \mathbb{R}^d | d(x,m) = r\}.$$

Hyperspherical regions centered at a point m with a boundary that has non-empty intersection with an object E play a central role in our notion of fatness. Therefore, the following definition is useful.

Definition 2.1 $[U_{m,E}, U_E]$
Let $m \in \mathbb{R}^d$ and let $E \subseteq \mathbb{R}^d$ be an object. The set $U_{m,E}$ is defined as:

$$U_{m,E} = \{S_{m,r} \subseteq \mathbb{R}^d \mid \partial S_{m,r} \cap E \neq \emptyset\}$$

The set U_E is defined as:

$$U_E = \bigcup_{m \in E} U_{m,E}$$

So, U_E is the set of all hyperspherical regions with center inside E that do not fully contain E. Figure 1 gives a two-dimensional example showing two circular regions S_0 and S_1 belonging to U_E and two circular regions S_2 and S_3 that do not belong to U_E. The region S_0 lies completely inside the object E and is therefore easily seen to be an element of U_E. The region S_1 is only partly covered by E but since its center lies inside the object E and its boundary has non-empty intersection with E, the region S_1 is a member of U_E. The circular region S_2 does not belong to U_E because its

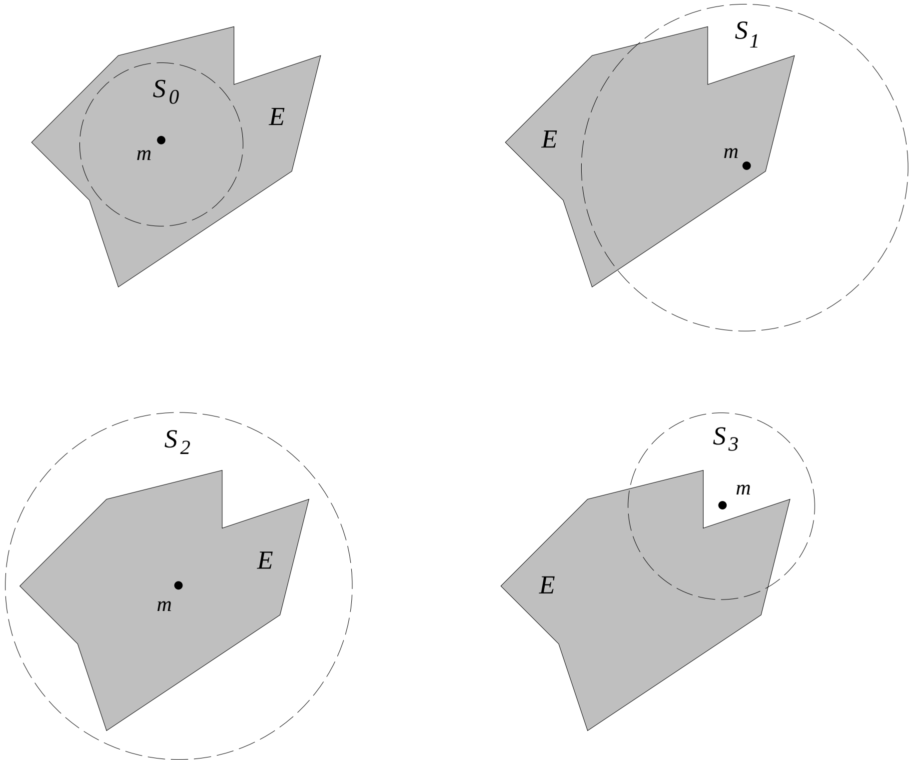

Figure 1: Illustration of the definition of U_E: $S_0, S_1 \in U_E$ and $S_2, S_3 \notin U_E$.

boundary has empty intersection with E, whereas S_3 is not a member of U_E because it has its center outside E.

We define fatness in a way such that objects are not only "compact" but also do not have extremely thin protuberances. The definition of fatness involves some positive number k. This number is a measure for the actual fatness of the object. If the value of k is increased then the object is allowed to be less fat. For objects with a boundary with infinitesimally thin protuberances (e.g. line segments) it is impossible to find such a k, so these objects can never be fat.

Definition 2.2 [***k*-fatness**]
Let $E \subseteq \mathbb{R}^d$ be an object and let k be a positive constant. The object E is k-fat if:

$$\forall S \in U_E \qquad k \cdot volume(E \cap S) \geq volume(S).$$

Informally, an object E is k-fat if the part of any hyperspherical region S (with a boundary that intersects E and its center inside E) covered by the object E is at least a $\frac{1}{k}^{\text{th}}$ of S. The choice for hyperspherical regions in the definition of fatness is rather arbitrary. In fact we could have used any compact region, e.g. hypercubic regions,

regions bounded by simplices etc. Any k-fat object according to one definition is easily seen to be k'-fat according to another definition for some k' that is only a constant multiple of k. Note that there is a straightforward property that an object that is k-fat is also k'-fat for $k' \geq k$.

The lower bound on the value of k differs from dimension to dimension. There are for example no 1-fat objects at all; there can be 5-fat obstacles in a two-dimensional workspace but 5-fat objects in a three-dimensional workspace do not exist. This is inherent to the definition of k-fatness. Suppose we have a k-fat object E with diameter δ. The volume of this object is bounded from above by the volume of a hypersphere with diameter δ (or radius $\delta/2$). The diameter of E is δ, so there is a pair of points on the boundary of E that are a distance δ apart; let $m, m' \in E$ be these two points. The hyperspherical region $S_{m,\delta}$ is an element of U_E since $m' \in \partial S_{m,\delta}$ and $m' \in E$. (Similarly, the hyperspherical region $S_{m',\delta}$ is an element of U_E.) Hence, the set U_E contains an element S with radius δ. We know that $volume(E \cap S) \leq volume(E) \leq C_d \cdot (\delta/2)^d$ and $volume(S) = C_d \cdot \delta^d$, where C_d is the dimension-dependent multiplier in the volume formulae for hyperspheres [1]. Combination with Definition 2.2 (E is k-fat and $S \in U_E$) yields $k \geq 2^d$. The boundary value 2^d-fatness is only obtained for hyperspherical objects; hyperspherical objects have maximal fatness.

The definition of k-fatness given in Definition 2.2 has a very "local" character: a certain portion of the proximity of every point in the object must be covered by the object too. As stated before, this locality prohibits objects with infinitesimally thin protuberances, even if these protuberances are extremely short. A huge spherical object with a very short line segment sticking out of its boundary will not be k-fat for any value of k. A more natural way to define fatness might be the more "global" type of fatness given in Definition 2.3. For convenience, we will refer to it as *thickness*. Here, we only compare the volume of the entire object to the volume of its minimal (volume) enclosing hypersphere: the volume of the object should be at least a certain portion of the minimal enclosing hypersphere of the object. This more liberal definition allows objects with small protuberances. If E is an object then we denote the minimal enclosing hypersphere of E by MES_E.

Definition 2.3 [**k-thickness**]
Let $E \subseteq \mathbb{R}^d$ be an object and let $k \geq 1$ be a constant. The object E is k-thick if:

$$k \cdot volume(E) \geq volume(MES_E).$$

The definition of k-thickness involves just one hypersphere instead of infinitely many. Note that not necessarily $MES_E \in U_E$: the minimal enclosing hypersphere of an object can have its center outside the object. Again we have the straightforward property that an object that is k-thick is also k'-thick for $k' \geq k$. Spherical objects are

[1]For even dimension $C_d = C_{2n} = \frac{\pi^n}{n!}$. For odd dimension $C_d = C_{2n+1} = \frac{2(2\pi)^n}{(2n+1)!!}$. See, e.g., [3, Section 394].

1-thick, because the minimal enclosing hyperspheres of such objects are the obstacles themselves.

Unfortunately, the notion of k-thickness does not result in low complexities of the free space. Hence, the definition is too general for our purposes. We could restrict ourselves to convex objects but, as we will see below, in that case thickness is equivalent to fatness. Therefore, we have chosen to use the definition of fatness stated as Definition 2.2 because it also allows for non-convex objects. The property of the set $U_{m,E}$ for a convex region E given in the next lemma is a useful tool in the proof of the equivalence of thickness and fatness for convex objects.

Lemma 2.4 *Let $E \subseteq \mathbb{R}^d$ be a convex object and $m \in E$. Let $S_{m,r} \in U_{m,E}$ and $S_{m,R} \in U_{m,E}$ with $r \leq R$. Now the following inequality holds:*

$$\frac{volume(E \cap S_{m,r})}{volume(S_{m,r})} \geq \frac{volume(E \cap S_{m,R})}{volume(S_{m,R})}.$$

Proof: We use a polar coordinate frame with origin m and angles $\phi, \theta_1, \ldots, \theta_{d-2}$, with $0 \leq \phi < 2\pi$ and $0 \leq \theta_1, \ldots, \theta_{d-2} \leq \pi$. Each combination of angles $(\phi, \theta_1, \ldots, \theta_{d-2})$ specifies a viewing direction from m. Since the object E is convex, each point on the boundary of E can be seen from m. Therefore, the relation between the viewing direction and the distance to the boundary of E is a function. The same obviously holds for both spheres. So, there are three functions $\rho_E, \rho_{S_{m,r}}, \rho_{S_{m,R}} : [0, 2\pi) \times [0, \pi]^{d-2} \longrightarrow \mathbb{R}^+ \cup \{0\}$, that give the distance from m to the boundary of E, $S_{m,r}$, and $S_{m,R}$ respectively. The latter two functions are constant: $\rho_{S_{m,r}}(\phi, \theta_1, \ldots, \theta_{d-2}) = r$ and $\rho_{S_{m,R}}(\phi, \theta_1, \ldots, \theta_{d-2}) = R$.
Let $f, F : [0, 2\pi) \times [0, \pi]^{d-2} \longrightarrow [0, 1]$ be defined as follows:

$$f(\phi, \theta_1, \ldots, \theta_{d-2}) = \min(\frac{\rho_E(\phi, \theta_1, \ldots, \theta_{d-2})}{r}, 1),$$

$$F(\phi, \theta_1, \ldots, \theta_{d-2}) = \min(\frac{\rho_E(\phi, \theta_1, \ldots, \theta_{d-2})}{R}, 1).$$

Integrating the product of function f to some power i and some determinant function Φ over all angular domains yields the ratio of $volume(E \cap S_{m,r})$ and $volume(S_{m,r})$ as given in the left-hand side of the inequality that is to be proven. The right-hand side is obtained by integrating the product of function F to the same power i and the same determinant function Φ. This Φ is a product of $(\sin \theta_i)^j$-terms. Since $0 \leq \theta_1, \ldots, \theta_{d-2} \leq \pi$, function Φ's range is restricted to $[0, 1]$. Function f and F have the same range. If we can prove that $f(\phi, \theta_1, \ldots, \theta_{d-2}) \geq F(\phi, \theta_1, \ldots, \theta_{d-2})$, for all $0 \leq \phi < 2\pi$ and $0 \leq \theta_1, \ldots, \theta_{d-2} \leq \pi$, then, because Φ, f, and F only have non-negative function values, the integral containing f will yield a larger value than the one containing F, and hence the inequality involving the volumes will be proved.

Relevant changes in the values of f and F obviously appear at $\rho_E(\phi, \theta_1, \ldots, \theta_{d-2}) = r$ and $\rho_E(\phi, \theta_1, \ldots, \theta_{d-2}) = R$. Therefore, we consider three different ranges for the value of $\rho_E(\phi, \theta_1, \ldots, \theta_{d-2})$.

1. If $\rho_E(\phi, \theta_1, \ldots, \theta_{d-2}) \leq r$ then:
   $f(\phi, \theta_1, \ldots, \theta_{d-2}) = \rho_E(\phi, \theta_1, \ldots, \theta_{d-2})/r \geq \rho_E(\phi, \theta_1, \ldots, \theta_{d-2})/R = F(\phi, \theta_1, \ldots, \theta_{d-2})$.
2. If $r \leq \rho_E(\phi, \theta_1, \ldots, \theta_{d-2}) \leq R$ then:
   $f(\phi, \theta_1, \ldots, \theta_{d-2}) = 1 \geq \rho_E(\phi, \theta_1, \ldots, \theta_{d-2})/R = F(\phi, \theta_1, \ldots, \theta_{d-2})$.
3. If $R \leq \rho_E(\phi, \theta_1, \ldots, \theta_{d-2})$ then:
   $f(\phi, \theta_1, \ldots, \theta_{d-2}) = 1 = F(\phi, \theta_1, \ldots, \theta_{d-2})$.

Figure 2 shows a two-dimensional example of each of the three cases given above.

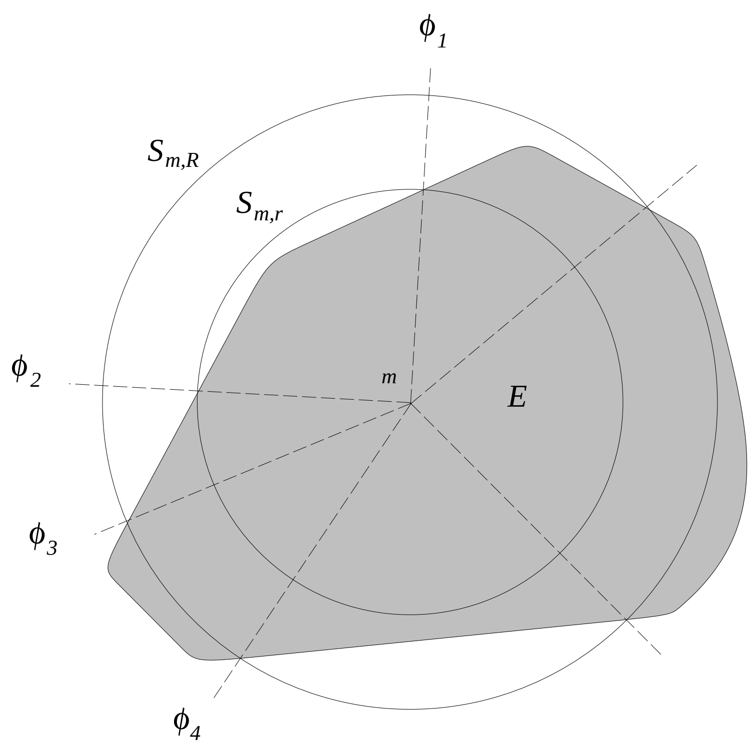

Figure 2: The angular interval $[\phi_1, \phi_2]$ is an example of case (1), interval $[\phi_2, \phi_3]$ is an example of case (2), and the angular interval $[\phi_3, \phi_4]$ is an example of case(3).

Combining the three different ranges, we obtain $f(\phi, \theta_1, \ldots, \theta_{d-2}) \geq F(\phi, \theta_1, \ldots, \theta_{d-2})$, for all $0 \leq \phi < 2\pi$ and $0 \leq \theta_1, \ldots, \theta_{d-2} \leq \pi$. $\square$

Lemma 2.4 shows that in each set $U_{m,E}$ the portion of a hyperspherical region that is covered by the object E does not increase as the radius of the hyperspherical region increases. The ratio is therefore minimal for region $\sup U_{m,E}$ (the region in $U_{m,E}$ with maximal volume). In the sequel we will use the abbreviation

$$ES_{m,E} = \sup U_{m,E}.$$

ES stands for enclosing sphere because $\sup U_{m,E}$ is actually the enclosing hyperspherical region of the object E centered at point m. A consequence of Lemma 2.4 is that if $k \cdot volume(E \cap ES_{m,E}) \geq volume(ES_{m,E})$ holds then we can conclude that $k \cdot volume(E \cap S) \geq volume(S)$ for all $S \in U_{m,E}$. Define the set ES_E of all enclosing hyperspherical regions centered at some point in the object:

$$ES_E = \bigcup_{m \in E} \{ES_{m,E}\}.$$

It is clear that $ES_E \subseteq U_E$. Due to Lemma 2.4 we can reformulate the requirement given in Definition 2.2 for convex objects. Note that for all $S \in ES_E$, the obvious equality $E \cap S = E$ holds. A convex object E is k-fat if:

$$\forall S \in ES_E \qquad k \cdot volume(E) \geq volume(S).$$

We are now ready to prove the equivalence of thickness and fatness of convex objects.

Theorem 2.5 *Let $E \subseteq \mathbb{R}^d$ be a convex object. Then*

$$E \text{ is } k\text{-fat} \Rightarrow E \text{ is } k'\text{-thick},$$

and

$$E \text{ is } l\text{-thick} \Rightarrow E \text{ is } l'\text{-fat},$$

where $k' = \mathcal{O}(k)$ and $l' = \mathcal{O}(l)$.

Proof:
E is k-fat $\Rightarrow$ E is k'-thick:

> Choose some hyperspherical region $S \in ES_E$. The object E is k-fat and $ES_E \subseteq U_E$, so $k \cdot volume(E) = k \cdot volume(E \cap S) \geq volume(S)$. Region S is some enclosing hyperspherical region of E and MES_E is defined as the minimal volume enclosing hyperspherical region of E, so obviously $volume(MES_E) \leq volume(S)$ holds. Combining both inequalities results in $k \cdot volume(E) \geq volume(MES_E)$, proving k'-thickness of E, with $k' = k$.

E is l-thick $\Rightarrow$ E is l'-fat:

> The convex object E is l-thick, so the inequality $l \cdot volume(E) \geq volume(MES_E)$ holds. By Lemma 2.4 and the convexity of E we know that it suffices to prove that $\forall S \in ES_E : \quad l' \cdot volume(E) \geq volume(S)$, for some constant l'. Let δ be the diameter of MES_E and let ϵ be the diameter of the object E. The obstacle E fits inside MES_E so trivially $\epsilon \leq \delta$. The diameter of the object E is determined by two points m and m' on its boundary. The

radius of a hyperspherical region in ES_E is at most ϵ. This is the radius of the largest regions $ES_{m,E}$ and $ES_{m',E}$.

We have $volume(MES_E) = C_d \cdot (\delta/2)^d$ and for all $S \in ES_E$: $volume(S) \leq C_d \cdot \epsilon^d$, where C_d is the dimension-dependent multiplication factor mentioned earlier in this section. Combination of all equalities and inequalities yields for all $S \in ES_E$:

$$\begin{array}{rl} & 2^d \cdot l \cdot volume(E) \\ \geq & 2^d \cdot volume(MES_E) \\ = & C_d \cdot \delta^d \\ \geq & C_d \cdot \epsilon^d \\ \geq & volume(S), \end{array}$$

proving l'-fatness of the convex object E, with $l' = 2^d \cdot l$. □

A consequence of Theorem 2.5 is that the complexity results that we prove for convex objects that are k-fat also hold for convex objects that are k-thick. In the rest of this paper we will only consider fatness, not thickness.

Let us consider some examples of fat shapes. Spherical objects are obviously fat, because we have shown earlier that a d-sphere has maximal achievable 2^d-fatness. Other shapes that are fat (in two-dimensional space) include squares, rectangles with bounded aspect ratio, and triangles with minimum angle restriction.

In [1] an approximate motion planning algorithm is given for a rectangular robot. The complexity of the algorithm decreases as the ratio of the rectangle's sides gets closer to 1. A rectangle with nonzero side lengths is fat according to our definition for some k depending on the aspect ratio. The value of k will increase as the rectangle becomes "narrow". The largest circular region in U_E (centered at a rectangle corner and enclosing the rectangle) has radius $\sqrt{a^2+b^2}$, so a rectangle with sides a and b is $\frac{\pi(a^2+b^2)}{ab}$-fat. Clearly, we get maximum fatness for squares and no fatness, i.e., there is no constant k such that the object is k-fat, if either $a = 0$ or $b = 0$.

In [11] a different notion of fatness is introduced for triangles by imposing a restriction on the angles in the triangle. A triangle is called δ-fat if each of its three internal angles is at least δ. A δ-fat triangle is also fat according to our definition. Assume that we are given a δ-fat triangle with a longest edge e. The triangle has minimum area if the other two angles have magnitudes δ and $\pi - 2\delta$. This minimal area is $\frac{1}{4}|e|^2 \tan \delta$. The largest circular region in U_E (centered at one of the end-points of e and enclosing the triangle) has radius $|e|$. Therefore, each δ-fat triangle is $\frac{4\pi}{\tan \delta}$-fat according to our fatness definition. Maximum fatness is obviously obtained for equilateral triangles and there is no fatness if $\delta = 0$. The latter triangle will also be non-fat in [11].

Many other classes of shapes are fat. In three-dimensional space we can think of cubes, boxes with bounded aspect ratio's in each of their bounding rectangular faces, equilateral tetrahedra etc.

3 The proximity of a k-fat object

In this section we consider the proximity of a k-fat object as a first step in finding an upper bound on the number of multiple contacts for a robot $\mathcal{B}$. An important observation is that two obstacles that are far (more than the diameter of the robot) apart can not be involved in any multiple contact for $\mathcal{B}$. Hence, obstacles that do cause such a contact must lie in each other's proximity.

A strategy for proving a linear upper bound on the number of multiple contacts for the robot $\mathcal{B}$ could be to prove that the number of multiple contacts involving a certain obstacle is only constant. Straightforward application of this strategy, however, would yield no result. If we have a situation with $n-1$ equally sized k-fat obstacles and one much larger k-fat obstacle, then considering the proximity of this large obstacle does not result in a constant upper bound on the number of obstacles that can participate in a multiple contact involving the large obstacle: all $n-1$ smaller obstacles might lie in the proximity of the large obstacle. Note, however that this strategy contains some redundancy: a multiple contact is counted more than once.

To avoid counting a single multiple contact more than once, we only count the number of larger obstacles that can participate in a multiple contact involving E. By starting with the smallest obstacle and repeatedly considering the next smallest obstacle we will count each multiple contact for the robot exactly once. It turns out that the number of such multiple contacts involving any obstacle E is constant.

Any k-fat obstacle E' that participates in some multiple contact with a given k-fat obstacle E must lie close to this obstacle E. Lemma 3.1 states that the number of k-fat objects that are larger than the k-fat object E lying in the proximity of E is bounded by a constant. In the lemma, the size of its minimal enclosing hypersphere is chosen as a measure for the size of an object. A different measure like diameter of the object, or volume of the object, would lead to the same result, but would make the proof less simple. Although here Lemma 3.1 is mainly a tool in proving the linear complexity result, it is also interesting in its own right.

Lemma 3.1 *Let $\mathcal{E} \subseteq \mathbb{R}^d$ be a set of k-fat objects and let $b > 0$ be a constant. Let $E \subseteq \mathcal{E}$ be an object and let δ be the diameter of its minimal enclosing hypersphere. Then the number of objects $E' \subseteq \mathcal{E}$ with a larger minimal enclosing hypersphere that lie within a distance $b \cdot \delta$ from E is at most $2^d \cdot k \cdot (b+1)^d$.*

Proof: The minimal enclosing hyperspherical region of E will be referred to as MES_E. Let S be the closed spherical region with a boundary that is concentric with the boundary of MES_E and having diameter $b \cdot \delta + \delta + b \cdot \delta = (1 + 2b) \cdot \delta$, and let T be the slightly larger closed spherical region with a concentric boundary and diameter $\delta/2 + b \cdot \delta + \delta + b \cdot \delta + \delta/2 = (2 + 2b) \cdot \delta$ (see Figure 3 for an example in the two-dimensional case).

Consider the object E, and let E' be an object with a larger minimal enclosing hypersphere having a distance less than $b \cdot \delta$ to E. Since all points with a distance less than $b \cdot \delta$ to E lie inside S, object E' must have non-empty intersection with S. Hence, there is a point $m \in E' \cap S$.

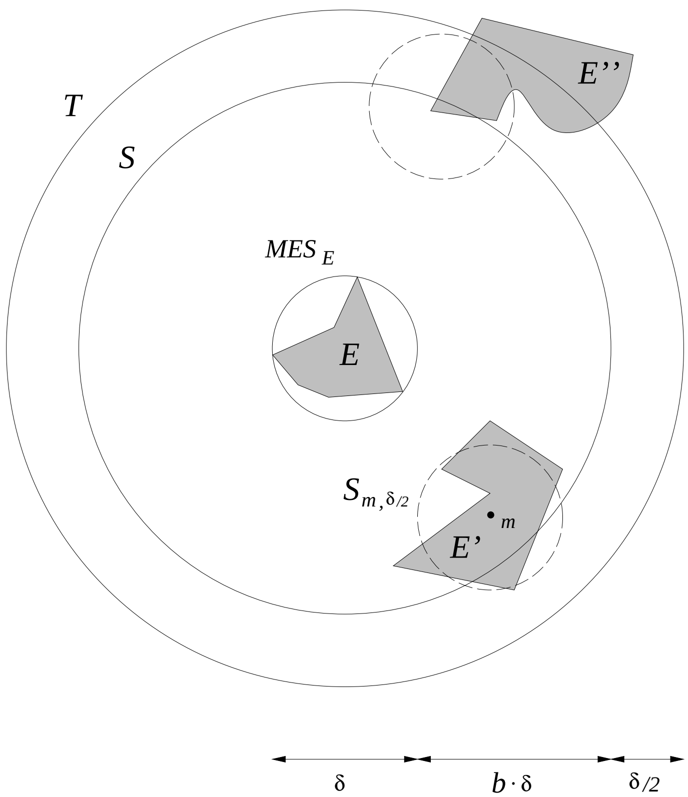

Figure 3: Illustration of the proof of Lemma 3.1.

As a next step we consider $S_{m,\delta/2}$, with $m \in E' \cap S$. Note that the size of $S_{m,\delta/2}$ is equal to the size of MES_E. In order to be able to give a lower bound on the volume of the object E' that is inside $S_{m,\delta/2}$, we prove that $S_{m,\delta/2} \in U_{E'}$. Obstacle E' has non-empty intersection with $S_{m,\delta/2}$, since the center $m \in E'$. Obstacle E' can not lie completely inside $S_{m,\delta/2}$, because then the assumption that E is the object with the smallest minimal enclosing hyperspherical region would be contradicted. So, the boundary of the hyperspherical region $S_{m,\delta/2}$ is intersected by E' and therefore the region $S_{m,\delta/2}$ is an element of $U_{E'}$.

The region $S_{m,\delta/2}$ clearly lies entirely inside the closed spherical region T. The object E' with a distance less than $b \cdot \delta$ to E has, because of its k-fatness and because $S_{m,\delta/2} \in U_{E'}$ for $m \in E' \cap S$, at least a part of volume $\frac{1}{k} \cdot volume(S_{m,\delta/2})$ inside region T, which is equal to $k^{-1} \cdot C_d \cdot (\delta/2)^d = 2^{-d}k^{-1} \cdot C_d \cdot \delta^d$, where C_d again is the dimension-dependent multiplier mentioned in Section 2. The volume of region T is $C_d \cdot (\frac{1}{2}(2+2b) \cdot \delta)^d = C_d \cdot (1+b)^d \cdot \delta^d$. Combining the lower bound on the volume that an object E' (with a distance less than $b \cdot \delta$ to E (and a minimal enclosing hypersphere larger than that of E) has inside T with the volume of the hyperspherical region T results in an upper bound of $2^d \cdot k \cdot (b+1)^d$ on the number of objects E' within a distance $b \cdot \delta$ from E. □

The actual magnitude of the constant upper bound on the number objects in the proximity of the smallest object is determined by three parameters: the dimension d, the fatness k of the objects, and the distance parameter b. For a fixed dimension d, the upper bound increases when the value of k or b increases. This is not surprising: the number of objects that can lie in the proximity of the smallest object increases when the objects become less fat (increasing k), or when a larger proximity is considered (increasing b).

4 Linear complexity of the free space

Let us now return to determining the complexity of the free space for some motion planning problem involving fat obstacles. The actual complexity of the free space depends on the number of intersections of hypersurfaces that bound the free space. Such a hypersurface is a set of placements of the robot $\mathcal{B}$ in which a certain feature of $\mathcal{B}$ is in contact with a certain feature of the boundary of $\mathcal{E}$. Intersections of hypersurfaces correspond to multiple contacts of the robot $\mathcal{B}$ with the boundary of $\mathcal{E}$. The intersections define faces on the hypersurfaces. The set of faces forms a description of the free space FP. If the number of possible multiple contacts of the robot $\mathcal{B}$ is low then the complexity of the free space is also low.

Before we focus on the problem of finding an upper bound on the number of multiple contacts, we first consider the notion of multiple contact itself. What kind of subspaces of the configuration space are defined by multiple contacts and how many obstacles can participate in a multiple contact?

The set of placements of the robot $\mathcal{B}$ in which a certain feature of $\mathcal{B}$ is in contact with a certain boundary feature of $\mathcal{E}$ of appropriate dimension forms an $(f-1)$-dimensional subspace (or hypersurface) in the f-dimensional configuration space. An intersection of two of these hypersurfaces corresponds to a simultaneous contact of the robot with two features of the boundary of the obstacle set $\mathcal{E}$. Such an intersection is an $(f-2)$-dimensional subspace of the configuration space. Moreover, a j-fold contact of the robot defines an $(f-j)$-dimensional space. Consequentially, the f-fold contacts appear at isolated points in the configuration space, and, hence, fix the position of the robot. Contacts that involve more than f obstacle features do not appear if we assume that the obstacles are in general position. (For a discussion of the issue of general position we refer to [5, 10].) Such contacts can therefore be discarded without affecting the complexity of the free space. We see that a robot $\mathcal{B}$ with f degrees of freedom can have up to f simultaneous contacts with the boundary of $\mathcal{E}$.

We consider the situation where a robot $\mathcal{B}$ moves amidst k-fat obstacles $E \subseteq \mathcal{E}$ in general position. The robot $\mathcal{B}$ is assumed to be not too big compared to the obstacles. Let the diameter of the smallest minimal enclosing hypersphere of any obstacle be δ_{min}. The diameter $\delta_{\mathcal{B}}$ of the robot is constrained by $\delta_{\mathcal{B}} \leq b \cdot \delta_{min}$, where b is some positive constant. This assumption regarding the size of the robot is not very restrictive: it basically rules out the situation where the robot $\mathcal{B}$ is so large that it would make the obstacles into point obstacles relative to its own size. The assumption will be satisfied in most practical

We assume that the number of features of the robot $\mathcal{B}$ is bounded by a constant and the number of features of the obstacle set $\mathcal{E}$ is n. As a consequence, the total number of hypersurfaces is $\mathcal{O}(n)$. The hypersurfaces are assumed to be algebraic of bounded degree, so that the intersection of j hypersurfaces consists of at most a constant number of connected components. This requirement for the degree of the hypersurfaces mainly means that the boundary of the robot and the obstacles must not be too irregularly shaped. As the hypersurfaces are of bounded degree this implies that the total complexity of the free space FP is bounded by $\mathcal{O}(n^f)$, where f is the dimension of the configuration space. The dimension of the configuration space equals the number of degrees of freedom of the robot. Each obstacle $E \subseteq \mathcal{E}$ is assumed to have only a constant number of features, so the number of obstacles is $\Omega(n)$.

The assumptions in the previous two paragraphs are sufficient to prove that the number of multiple contacts for a robot $\mathcal{B}$ is at most linear in the number of obstacles. We summarize these assumptions below.

- The workspace W of the robot $\mathcal{B}$ is the d-dimensional Euclidean space ($\mathbb{R}^d$).
- The workspace W of the robot $\mathcal{B}$ contains n k-fat obstacles $E \subseteq \mathcal{E} \subseteq \mathbb{R}^d$ in general position.
- The diameter $\delta_{\mathcal{B}}$ of the robot $\mathcal{B}$ is bounded: $\delta_{\mathcal{B}} \leq b \cdot \delta_{min}$, where $b > 0$ and δ_{min} is the diameter of the smallest minimal enclosing hypersphere of any obstacle $E \subseteq \mathcal{E}$.
- The robot $\mathcal{B}$ has constant complexity.
- Each obstacle $E \subseteq \mathcal{E}$ has constant complexity.
- The hypersurface in the configuration space corresponding to the set of robot placements in which a certain robot feature is in contact with a certain obstacle feature is algebraic of bounded degree.

The proximity result given in Lemma 3.1 is the key to successful application of the proof strategy presented in the previous section. Using that strategy we repeatedly consider an obstacle E and count the number of multiple contacts for the robot $\mathcal{B}$ involving E and obstacles with larger minimal enclosing hyperspheres. Lemma 3.1 guarantees that we find a constant upper bound on this number for each obstacle E. The resulting overall number of multiple contacts will be linear, which is stated in Theorem 4.1.

Theorem 4.1 *Let $\mathcal{E} \subseteq \mathbb{R}^d$ be a set of n k-fat obstacles of constant complexity. The diameter of the minimal enclosing hypersphere of each obstacle $E \subseteq \mathcal{E}$ is at least δ_{min}. Let $\mathcal{B}$ be a robot of constant complexity with f degrees of freedom and with diameter $\delta_{\mathcal{B}} \leq b \cdot \delta_{min}$. For each j $(2 \leq j \leq f)$, the number of j-fold contacts of the robot $\mathcal{B}$ is linear in the number of obstacles: $\mathcal{O}(n)$.*

Proof: Consider some obstacle $E \subseteq \mathcal{E}$. The diameter δ of the minimal enclosing hypersphere of E is at least δ_{min}. We count the number of j-fold contacts of $\mathcal{B}$ that involve E and obstacles E' with a larger minimal enclosing hypersphere. Such an obstacle E' must lie within a distance $\delta_{\mathcal{B}}$ from E in order to allow $\mathcal{B}$ to touch E and E' simultaneously, and, hence, that E and E' both participate in a single j-fold contact for $\mathcal{B}$. Let p be the number of obstacles E' that lie within a distance $\delta_{\mathcal{B}}$ from E. Since $\delta_{\mathcal{B}} \leq b \cdot \delta_{min} \leq b \cdot \delta$, we know by Lemma 3.1 that p is bounded by the constant $2^d \cdot k \cdot (b+1)^d$.

A single j-fold contact is determined by j different pairs, each pair consisting of a robot feature and an obstacle feature. It is not determined by the robot and j different obstacles, because e.g. more than one feature of a single obstacle can be involved in a single j-fold contact. Let us assume that the robot has $x_{\mathcal{B}}$ different features and that the number of features of each obstacle E is bounded by $x_{\mathcal{E}}$.

The first contact is a contact between a robot feature and a feature of the obstacle E. Since the robot $\mathcal{B}$ and the obstacle E have $x_{\mathcal{B}}$ and $x_{\mathcal{E}}$ features respectively, we have at most $x_{\mathcal{B}} \cdot x_{\mathcal{E}}$ choices for this first contact. For each of the $j-1$ next contacts we can choose the obstacle feature on each of the p obstacles in the proximity of E, which gives a total number of $x_{\mathcal{E}} \cdot p$ possibly involved obstacle features. For each contact we can again choose from all $x_{\mathcal{B}}$ robot features. An upper bound on the number of choices for the remaining $j-1$ contacts is therefore at most $(x_{\mathcal{B}} \cdot x_{\mathcal{E}} \cdot p)^{j-1}$. Hence, the total number of j-fold contacts involving E is bounded by $(x_{\mathcal{B}} \cdot x_{\mathcal{E}} \cdot p)^{j-1} + x_{\mathcal{B}} \cdot x_{\mathcal{E}}$, which is a constant.

Subsequentially, we repeatedly choose a next obstacle E'' and again count the number of j-fold contacts of $\mathcal{B}$ that involve E'' and obstacles with a larger minimal enclosing hypersphere. The counting for the obstacle E'' obviously also results in at most $(x_{\mathcal{B}} \cdot x_{\mathcal{E}} \cdot p)^{j-1} + x_{\mathcal{B}} \cdot x_{\mathcal{E}}$ j-fold contacts.

Adding all the n constant upper bounds results in an overall upper bound on the number of j-fold contacts of $n \cdot ((x_{\mathcal{B}} \cdot x_{\mathcal{E}} \cdot p)^{j-1} + x_{\mathcal{B}} \cdot x_{\mathcal{E}})$, which is $\mathcal{O}(n)$, since $x_{\mathcal{B}}$, $x_{\mathcal{E}}$, p, and j are constants. □

Note that the value of j in Theorem 4.1 ranges from 2 to f. The number of single contacts is also linear; a different hypersurface corresponds to each contact between a robot feature and an obstacle feature, giving a total of $\mathcal{O}(n)$ hypersurfaces since there are n obstacles and because of the constant complexity of the robot and each obstacle.

The $(f-j)$-dimensional subspace defined by a single j-fold contact is not necessarily connected. Figure 4 shows an example for $f = 3$ and $j = 2$, where it is impossible for the robot to move from Z_0 to Z_1 without losing contact with either the upper or the lower obstacle feature. The 1-dimensional subspace induced by the contact with both features is therefore non-connected. Our assumption that all contact

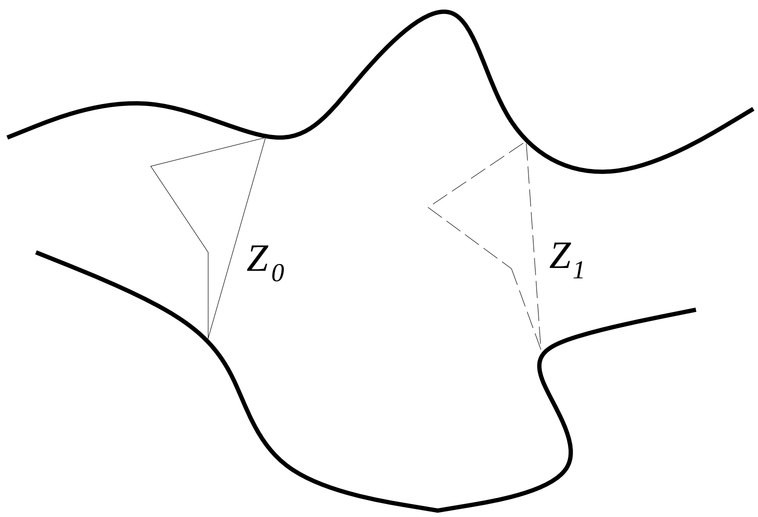

Figure 4: There is no continuous motion of the robot from Z_0 to Z_1 during which it remains in contact with both features.

hypersurfaces are of bounded degree, however, implies that the number of different connected subspaces induced by a single multiple contact is bounded by some small constant. The complexity of the free space is now solely determined by the number of multiple contacts, since the contibution of a single multiple contact to the free space apparantly has constant complexity. Variable j in Theorem 4.1 can only have $f-1$ different values, so the total number of multiple contacts is linear, and, hence, the free space has linear complexity.

Corollary 4.2 *Let $\mathcal{E} \subseteq \mathbb{R}^d$ be a set of n k-fat obstacles of constant complexity. The diameter of the minimal enclosing hypersphere of each obstacle $E \subseteq \mathcal{E}$ is at least δ_{min}. Let $\mathcal{B}$ be a robot of constant complexity with f degrees of freedom and with diameter $\delta_{\mathcal{B}} \leq b \cdot \delta_{min}$. The free space for the robot $\mathcal{B}$ moving amidst the k-fat obstacles of set $\mathcal{E}$ has linear complexity.*

The constant that we obtained in Theorem 4.1 can be quite high: the first contact for the robot is a feature of E, but each of the other $j-1$ contacts are chosen from all features in the proximity of E. In practice, this approach clearly yields a bound that is far from tight for a number of reasons. A robot touching a feature w of E might not be able to touch all features in the proximity of E because some of them are just too far away. Moreover, the robot $\mathcal{B}$ will not be able to touch a large number of features that lie close enough to w but cannot be touched by $\mathcal{B}$, simply because $\mathcal{B}$'s shape does not allow it. By fixing a certain contact for $\mathcal{B}$, some features will stop being candidates for being involved in the j-fold contact for $\mathcal{B}$, because of either one of both reasons. Hence, fixing a contact reduces the set of candidates for the remaining contacts. Clearly, the number of actual j-fold contacts for $\mathcal{B}$ will remain far below the upper bound of Theorem 4.1.

5 Conclusion

In this paper we have proposed a new definition of fatness of geometric objects and compared it to alternative definitions. We have shown that at most a constant number of larger fat objects can lie in the proximity of a given fat object. Finally, we used this result to prove that, under some realistic assumptions, the complexity of the free space of a robot $\mathcal{B}$ moving amidst k-fat obstacles is linear. The assumptions include constant complexity requirements for the robot and the obstacles, and a bound on the size of the robot relative to the size of the obstacles. In most practical situations, all these constraints will be satisfied. The assumption on the size of the robot basically forbids the case of a large robot moving amidst extremely small obstacles. The constant complexity requirement and an additional bound on the degree of the hypersurfaces defined by robot-obstacle contacts can be translated into a requirement that the boundary of the obstacles and the robot do not have a too complex shape. Both assumptions are common assumptions in motion planning.

If an obstacle does not meet the complexity assumption, the linear complexity result presented in this paper can still be applicable. In that case an adequate constant complexity outer approximation of the obstacle might be found with a volume that is not too much larger than the volume of the obstacle itself. If we can find such an outer approximation then we can do motion planning for this approximation at the cost of a relatively small reduction of the number of solutions. A constant complexity outer approximation is certainly superior to a simple approximation like a minimal enclosing hypersphere with respect to minimizing the reduction of the solution space. A similar procedures can be applied if the complexity of the robot is too high.

The linear complexity result leads to better time bounds for a specific class of motion planning algorithms for a robot moving amidst fat obstacles. This class consists of algorithms with a running time that depends on the complexity of the free space. The motion planning algorithm of Sifrony and Sharir [17] for a ladder moving in two-dimensional space is an example of such an algorithm, because its running time is determined by the number of feature pairs that are less than the length of the ladder apart, which, on its turn, is among the factors that determine the complexity of the free space. An immediate consequence of our results is that the number of such feature pairs is linear. Hence, we obtain an algorithm with improved running time in case of fat obstacles. Algorithms that consider *all* features obviously do not benefit from our result. In [18] we will present an efficient motion planning algorithm with a running time that depends on the complexity of the free space for a robot moving amidst fat obstacles.

References

[1] H. Alt, R. Fleischer, M. Kaufmann, K. Mehlhorn, S. Näher, S. Schirra, and C. Uhrig, Approximate Motion Planning and the Complexity of the Boundary of the Union of Simple Geometric Figures, *Proc. 6th Symp. on Computational Geometry* (1990), pp. 281-289.

[2] F. Avnaim, J.D. Boissonnat, and B. Faverjon, A Practical Exact Motion Planning Algorithm for Polygonal Objects Amidst Polygonal Obstacles, Rapports de Recherche 890, INRIA, France (1988).

[3] G.M. Fikhtengol'ts, *The Fundamentals of Mathematical Analysis, Vol. II*, English edition, Pergamon Press (1965).

[4] L. Guibas and M. Sharir, Combinatorics and algorithms of arrangements, to appear in *New Trends in Discrete and Computational Geometry*, (J. Pach, Ed.).

[5] D. Halperin, *Algorithmic Motion Planning via Arrangements of Curves and of Surfaces,* Ph.D. Dissertation, Computer Science Department, Tel Aviv University, July 1992.

[6] D. Halperin, M.H. Overmars, and M. Sharir, Efficient motion planning for an L-shaped object, *SIAM Journal on Computing* **21** (1992), pp. 1-23.

[7] K. Kedem and M. Sharir, An Efficient Motion-Planning Algorithm for a Convex Rigid Polygonal Object in Two-Dimensional Polygonal Space, *Discrete Comput. Geom.* **5** (1990), pp. 43-75.

[8] J.C. Latombe, *Robot Motion Planning*, Kluwer Academic Publishers, Boston (1991).

[9] D. Leven and M. Sharir, An Efficient and Simple Motion Planning Algorithm for a Ladder Amidst Polygonal Barriers, *Journal of Algorithms* **8** (1987), pp. 192-215.

[10] D. Leven and M. Sharir, On the number of critical free contacts of a convex polygonal object moving in two-dimensional polygonal space, *Discrete and Computational Geometry* **2** (1987), pp. 255–270.

[11] J. Matoušek, N. Miller, J. Pach, M. Sharir, S. Sifrony, and E. Welzl, Fat Triangles Determine Linearly Many Holes, *Proc. 32nd IEEE Symp. on Foundations of Computer Science* (1991), pp. 49-58.

[12] C. Ó'Dúnlaing and C.K. Yap, A "Retraction" Method for Planning the Motion of a Disc, *Journal of Algorithms* **6** (1985), pp. 104-111.

[13] C. Ó'Dúnlaing, M. Sharir, and C.K. Yap, Retraction: A new approach to motion planning, *Proc. 15th ACM Symp. on the Theory of Computing* (1983), pp. 207-220.

[14] M.H. Overmars, Point Location in Fat Subdivisions, Technical Report RUU-CS-91-40, Department of Computer Science, Utrecht University (1991).

[15] J.T. Schwartz and M. Sharir, On the Piano Movers' Problem I. The case of a two-dimensional rigid polygonal body moving amidst polygonal boundaries, *Comm. Pure Appl. Math.* **36** (1983), pp. 345-398.

[16] M. Sharir, Algorithmic motion planning in robotics, *Computer* **22**, March 1989, pp. 9–20.

[17] S. Sifrony and M. Sharir, A New Efficient Motion Planning Algorithm for a Rod in Two-Dimensional Polygonal Space, *Algorithmica* **2** (1987), pp. 367-402.

[18] A.F. van der Stappen, D. Halperin, and M.H. Overmars, An Efficient Motion Planning Algorithm for a Robot Moving Amidst Fat Obstacles, to appear.

[19] C.-K. Yap, Algorithmic motion planning, *Advances in Robotics, Vol. I: Algorithmic and Geometric Aspects of Robotics*, (J.T. Schwartz and C.-K. Yap, Eds.), Lawrence Erlbaum Associates, New Jersey, 1987, pp. 95–143.