Constructing Resultant Polytopes by a Vertex Oracle

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Outline

1. Resultant polytope
   - Algebraic motivation
   - Geometric definition
2. Algorithm
   - Approach
   - Implementation techniques
3. Current work
   - Enumerate resultant polytopes
   - Discriminant

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Polytope what?

Example

$$f_0 = c_{00} + c_{01}x, \quad f_1 = c_{10} + c_{11}x,$$

$$\text{resultant } \mathcal{R} = c_{00}c_{11} - c_{01}c_{10} = \det \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix}$$

CH of $(1,0,0,1)$, $(0,1,1,0)$ is 1-dimensional.

Resultant polytope

Definition

The (sparse) resultant $\mathcal{R}$ (aka eliminant) of polynomials $f_0, \ldots, f_n$, where

$$f_i = \sum_{a \in A_i} c_{ia} x^a, \quad x^a = x_1^{a_1} \cdots x_n^{a_n},$$

with symbolic $c_{ia} \neq 0$ and exponents $A_i \subset \mathbb{Z}^n$, is the unique irreducible integer polynomial in the c_{ij}, which vanishes iff the f_i have a common complex root.

It generalizes ...

...the determinant of an (overconstrained) linear system,
...the Sylvester resultant of two univariate polynomials $(n = 1)$

Problem definition

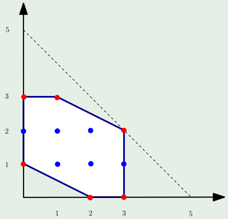
Resultant polytope

Given pointsets $A_0, \ldots, A_n \subset \mathbb{Z}^n$, compute the Newton polytope $N(\mathcal{R}) \subset \mathbb{R}^{\sum_i |A_i|}$, i.e. the convex hull of exponent vectors in $\mathcal{R}$ with nonzero coefficient.

For linear systems, the Birkhoff polytope is the Newton polytope of the determinant.

Example

$$f(x,y) = 8y + xy - 24y^2 - 16x^2 + 220x^2y - 34xy^2 - 84x^3y + 6x^2y^2 - 8xy^3 + 8x^3y^2 + 8x^3 + 18y^3$$



Newton polytopes capture the notion of total degree in sparse elimination theory

Bivariate example

Example

$$n = 2$$

$$f_0 := c_{00} - c_{01}x_0x_1,\ f_1 := c_{10} - c_{11}x_0x_1^2,\ f_2 := c_{20} - c_{21}x_0^2.$$

Then, for symbolic (hence generic) c_{ij},

$$x_0^4x_1^4 = \frac{c_{00}^4}{c_{01}^4},\ x_0^2x_1^4 = \frac{c_{10}^2}{c_{11}^2},\ x_0^2 = \frac{c_{20}}{c_{21}}.$$

Hence the sparse resultant is

$$\mathcal{R} = -c_{00}^4c_{11}^2c_{21} + c_{01}^4c_{10}^2c_{20},$$

and $N(\mathcal{R})$ is a segment in $\mathbb{R}^6$.

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Constructive definition

Theorem (Sturmfels'94)

There exists a surjection ρ from the regular fine mixed subdivisions S of $A_0 + \cdots + A_n$ onto the vertices of $N(\mathcal{R})$, where

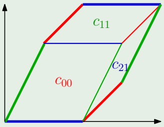
$$\rho_S(a) = \sum_{a\text{-mixed } \sigma \in S:\ a \in vtx(\sigma)} \mathrm{vol}(\sigma).$$

Subdivisions mapping to the same $N(\mathcal{R})$ vertex are ρ-equivalent.

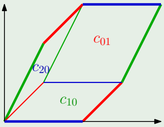
Moreover, the corresponding lifting vector $\in$ normal cone of vertex.

Example

All cells mixed:



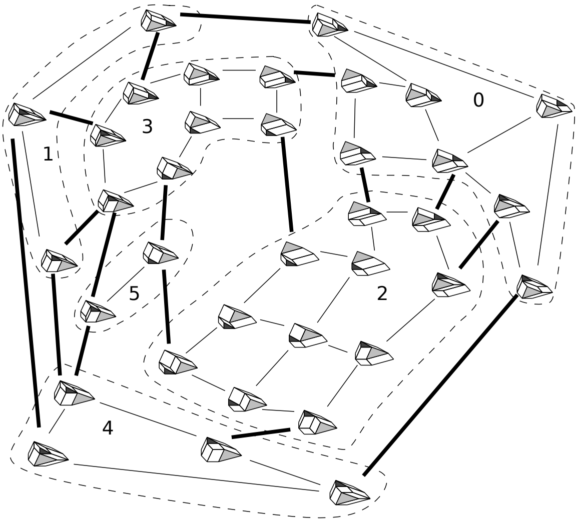
$c_{00}^4 c_{11}^2 c_{21}$

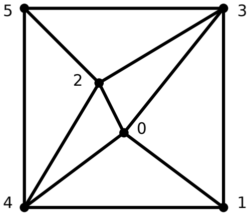


$4c_{01}^4 c_{10}^2 c_{20}$

$$\rho_S = (4, 0, 0, 2, 0, 1)$$
$$\rho_{S'} = (0, 4, 2, 0, 1, 0)$$
$$N(\mathcal{R}) \subset \mathbb{R}$$

$\max N(\mathcal{R}) \subset \mathbb{R}^3$





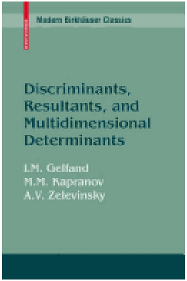
(Secondary) graph of mixed subdivisions for $|\mathcal{A}| = 3 + 2 + 3$ points $\in \mathbb{Z}^2$: the 6 equivalence classes are dotted; $N(\mathcal{R}) \subset \mathbb{R}^3$.

Cayley trick

Theorem (GKZ)

The following correspondence holds:

$A_0, \ldots, A_n \subset \mathbb{Z}^n$	$\mathcal{A} = \bigcup_{i=0}^{n} (A_i \times \{e_i\}) \subset \mathbb{Z}^{2n}$
subdivision of $\sum_i A_i$ *in Minkowski cells*	*polyhedral subdivision of* $\mathcal{A}$
regular	*regular*
fine (or tight) subdivision	*triangulation*
(regular fine) mixed subdivision	*regular triangulation*



Cayley set

Example

$f_0 = c_{00} - c_{01}x_1x_2 + c_{02}x_2, \; f_1 = c_{10} - c_{11}x_1x_2^2 + c_{12}x_2^2, \; f_2 = c_{20} - c_{21}x_1^2 + c_{22}x_2,$

$$\mathcal{A} = \left|\begin{array}{ccccccccc} 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 1 \\ \color{red}0 & \color{red}1 & \color{red}0 & \color{red}0 & \color{red}1 & \color{red}0 & \color{red}0 & \color{red}1 & \color{red}0 \\ \color{red}0 & \color{red}0 & \color{red}1 & \color{red}0 & \color{red}0 & \color{red}1 & \color{red}0 & \color{red}0 & \color{red}1 \end{array}\right| \begin{array}{c} A_i \\ \\ e_i \end{array}$$

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Overall approach

Algorithm

- Mixed subdivision (= regular triangulation) yields $N(\mathcal{R})$ vertex
- Incremental Beneath-Beyond: use facet normals as lifting vectors
- Lemma 5: Can refine regular subdivision into triangulation

Correctness

Given facet normal, compute the vertex extremal in its direction:

- if coplanar with the facet, it is an $N(\mathcal{R})$ facet
- else, add vertex to current polytope

Features and extensions

Features

- Lemmas 7, 8: Vertex oracle called $\#\text{vtx}(N(\mathcal{R})) + \#\text{fct}(N(\mathcal{R}))$ times
- Inner (current) and outer approximations of $N(\mathcal{R})$
- Also for orthogonal projections $\pi(N(\mathcal{R}))$ (pad normal vector with 0's)
- faster than tropical geometry [Yu-Jensen'11] for $dim(N(\mathcal{R})) \leq 6$

Further

- Secondary polytopes (secondary vertex from regular triangulation)
- V-oracle paradigm: bottleneck is going from V- to H-rep
- Volume approximation (Vissarion visits Bernd)
- Fast polytope approximation and Lattice points

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Implementation

- C++ CGAL-based software ResPol (respol.sourceforge.net)
- package `extreme_points_d` [Gärtner] preprocesses A_i
- experimental pack `triangulation` [Boissonnat,Devillers,Hornus:SoCG'09]

- `triangulation` for regular subdivision / triangulation of $\mathcal{A} \subset \mathbb{Z}^{2n}$
- placing triangulations of $\mathcal{A}$; non-projection coordinates correspond to points lifted to 0, placed once

triangulation package

- triangulation for representing (triangulated) $N(\mathcal{R}) \subset \mathbb{R}^{\sum |A_i|}$
- triangulation updates polytope with new vertex; latter always external (fast point location: know a red facet)
- triangulation maintains V- and H-representation of $N(\mathcal{R})$; triangulated (without extra charge)
- cells of triangulated facets have same normal: STL set ensures one per facet

Hash the determinants

What

- computing *similar* determinants for orientation, volume
- with cofactor expansion, we expand along last row (lifting)

How

- Hash a minor the first time, never compute it again.
- speeds up 18-100x

Next

- Storage issue: clean up hashtable from time to time.
- General BB algorithms [Fisikopoulos-Peñaranda,ESA'12]

CGAL support

Observe

- Fixing dim $\mathcal{A}$ at compile time yields $< 1\%$ speedup
- Filtered kernel similarly insignificant

Future

- faster incremental regular triangulations to exploit 0's in lifting vector
- f-vectors (`polymake`)
- low-dimensional simplices, e.g. volume
- towards high-dimensional CGAL

Outline

A particular family

Definition

Given $A_0, \dots, A_n \subset \mathbb{Z}^n$ and direction w, define a *subsystem* by

$$A_i' = \mathrm{face}_w(A_i) \subset \mathbb{Z}^k, \; k < n,$$

and consider the corresponding *polynomial subsystem*. Its solvability can be expressed by the resultant of the polynomials of some $A_0', \dots, A_k'$.

Theorem

Every face of $N(\mathcal{R})$ is the Minkowski sum of resultant polytopes of such subsystems.

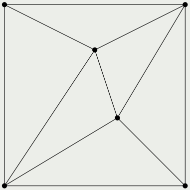
Enumeration

Theorem (Sturmfels)

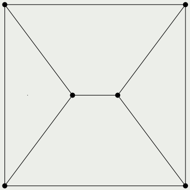
$N(\mathcal{R}) \subset \mathbb{R}^2 \Rightarrow$ *triangle.*
$N(\mathcal{R}) \subset \mathbb{R}^3 \Rightarrow$ *tetrahedron, square-based pyramid, first below.*
In any dimension, one $N(\mathcal{R})$ *is the simplex.*

Corollary

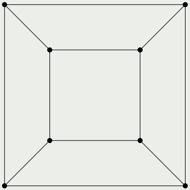
In $\mathbb{R}^4$, *for sufficiently generic pointsets, here are the* $3d$ *facets:*



max $3d$ $N(\mathcal{R})$



segment+triangle



sum of 3 segments

f-vectors

$$\begin{array}{cccc}
(6, & 15, & 18, & 9) \\
(8, & 20, & 21, & 9) \\
(9, & 22, & 21, & 8) \\
\vdots & & & \\
(17, & 48, & 45, & 14) \\
(17, & 48, & 46, & 15) \\
(17, & 48, & 47, & 16) \\
(17, & 49, & 47, & 15) \\
(17, & 49, & 48, & 16) \\
(17, & 49, & 49, & 17) \\
(17, & 50, & 50, & 17) \\
(18, & 51, & 48, & 15) \\
(18, & 51, & 49, & 16) \\
(18, & 52, & 50, & 16) \\
(18, & 52, & 51, & 17) \\
(18, & 53, & 51, & 16) \\
(18, & 53, & 53, & 18)
\end{array}$$

$$\begin{array}{cccc}
(18, & 54, & 54, & 18) \\
(19, & 54, & 52, & 17) \\
(19, & 55, & 51, & 15) \\
(19, & 55, & 52, & 16) \\
(19, & 55, & 54, & 18) \\
(19, & 56, & 54, & 17) \\
(19, & 56, & 56, & 19) \\
(19, & 57, & 57, & 19) \\
(20, & 58, & 54, & 16) \\
(20, & 59, & 57, & 18) \\
(20, & 60, & 60, & 20) \\
(21, & 62, & 60, & 19) \\
(21, & 63, & 63, & 21) \\
(22, & 66, & 66, & 22)
\end{array}$$

Open

Almost symmetric f-vector?

4-dimensional $N(\mathcal{R})$

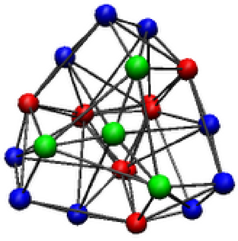
Theorem (Dickenstein,E,Fisikopoulos)

The 4d $N(\mathcal{R})$ are: the simplex, polytopes of 2 univariate polynomials [GKZ], Max (below), and smaller polytopes.

In progress

The max $N(\mathcal{R}) \subset \mathbb{R}^4$ has *f*-vector $(22, 66, 66, 22)$.

Prove, by counting subsystems:
there are $\leq 9 + 9 + 4 = 22$ facets,
and ≤ 66 ridges (shown).



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Discriminant

Definition

Discriminant The discriminant Δ of polynomials $f_1, \ldots, f_n$ in $n \geq 1$ variables with symbolic coefficients is the unique irreducible integer polynomial in the coefficients, which vanishes iff the f_i have a common double root.

It generalizes ...

...the case $n = 1$, e.g. $\Delta(ax^2 + bx + c) = b^2 - 4ac$,
...the resultant of an $(n+1) \times n$ system as $\Delta(F) :\ F$ has exponent set $\mathcal{A}$.

Discriminant polytope

To compute the Newton polytope $N(\Delta)$ we can construct a V-oracle which computes every $N(\Delta)$ vertex exactly only once.
The oracle is based on regular triangulations: computes a signed sum of volumes of all simplices.

Minkowski sums

Motivation

The discriminant polytope is expressed as an alternating Minkowski sum of resultant polytopes, whose pointsets $\mathcal{A}$ are rather easy to compute.

V-oracle

Suppose polytope P equals a (signed) Minkowski sum of polytopes P_i:

$$P = \sum_i s_i P_i, \quad s_i \in \{-1, +1\},$$

such that there exist V-oracles for all P_i. Then we can construct an efficient V-oracle for P, which computes every vertex of P exactly once. In particular, given direction w, let $v_i \in P_i$ be the corresponding w-extremal vertex, then

$$v = \sum_i s_i v_i$$

is the vertex of P extremal in the direction of w.

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Thank you!